CLD

§1.8 – §1.10

4 Oct 2025

[CLD, §1.8 – §1.10]

§1.8. cells & polyhedra, PL subspaces of $\mathbb{R}^n$.

§1.9. diff. forms on PL subspaces

§1.10. positive forms + $\alpha = \alpha_1 - \alpha_2$ on PL subsp.

positive

=

§1.8. Affine spaces $\mathbb{R}^n = V$

K-linear str. $\quad K \subseteq \mathbb{R}$ subfield.

implicitly : $\vec{V}_K \subseteq \vec{V}$ $\quad$ K-str.

K-vs

Affine maps : $f : \mathbb{R}^n \to \mathbb{R}^m$

$$f(x) = Ax + b$$

$$A \in M_{n \times m}(K) \quad b \in \mathbb{R}^m.$$

Often :

$K = \mathbb{Q}$.

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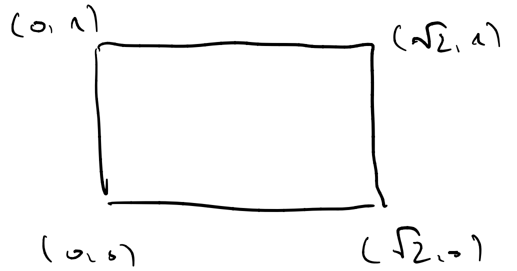
cells & polyhedra $\subseteq \mathbb{R}^n$

cell $= \{ f_1 \le 0, \dots, f_n \le 0 \} \subseteq \mathbb{R}^n$

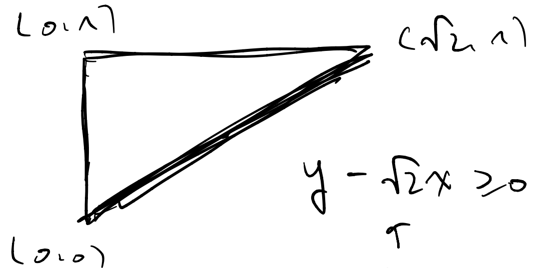
f_i affine form.

e.g.



cell.

$x - \sqrt{2} \le 0$



(not $\mathbb{Q}$-cell)

not cell

$y - \sqrt{2}x \geq 0$
$\uparrow$
$\notin \mathbb{Q}$

polyhedron $= \bigcup_{\text{finite}}$ cells (← "polytope" in [QD1, v1])

$\Big\downarrow$ bounded / compact

polytope

$f : \overset{\text{polyhedron}}{P} \longrightarrow \mathbb{R}^n$ is <u>polyhedral</u> if

$\exists\ P = \bigcup_{\text{finite}} C_i$ s.t. $f|_{C_i}$ affine.

<u>dimension</u> : $\dim C := \dim \langle C \rangle$

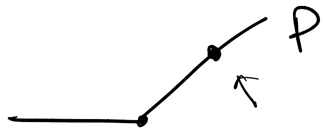
$\dim P := \sup_{C \subseteq P} \dim C = \max_i \dim C_i$

$P = \bigcup_{\text{finite}} C_i$

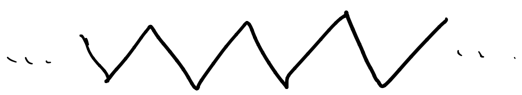
$$\dim_x P = \inf_{x \in \mathring{Q} \subseteq Q \subseteq P} \dim Q$$
$$= \dim Q, \quad Q \ni x \text{ small enough}$$

<u>tangent space</u>:

$$T_x P = \langle U \rangle$$



$$x \in U \subseteq P$$
U small enough



piecewise linear space = **PL**

paralinear

<u>Idea</u>: PL ~~subspace~~ = loc. finite union of cells.

<u>Precise definition</u>:

① <u>Def</u>. ($\mathcal{G}$-covering)

X top space,

$(X_i)_{i \in I}$ subspaces of X.

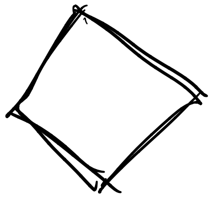
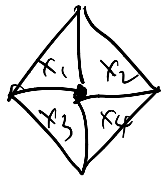
$(X_i)_{i \in I}$ is a <u>$\mathcal{G}$-covering</u> of X

if $\forall x \in X$. $\exists$ finite subfamily $(X_i)_{i \in I_x}$

st. 1) $x \in \bigcap_{i \in I_x} X_i$

2) $x \in \underbrace{\bigcup_{i \in I_x} X_i}_{\text{nbhd in } X} \subseteq X$ (openness)

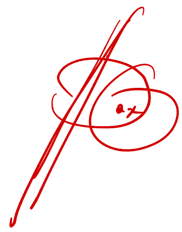
e.g.

$X =$  

is G-covering

[Berkovich, 1993, IHES] quasinet. §1.1.

e.g.
- Open coverings $\Rightarrow$ G-coverings
- locally finite family of cells C_i (closed) $\Rightarrow$ G-covering.
  s.t. $X = \bigcup_i C_i$



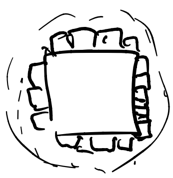
(2) $X \subseteq \mathbb{R}^n$ subspace.

Def. X is a PL subspace of $\mathbb{R}^n$

if $(C)_{\substack{C \text{ cells} \\ C \subseteq X}}$ is G-covering of X.

cf. cells, polyhedra – closed

Ex.
- cells / polyhedra are PL.
- open balls are PL
- open subset ✓
- PL $\cap$ PL $=$ PL $\Rightarrow$ open subset of PL ✓



Rmk. locally finite union of closed $\Rightarrow$ PL.

Rmk. X is PL $\Leftrightarrow$ $\exists\ (C_i)$ C_i cells $\subseteq X$ G-covering X.

$\Leftrightarrow$ $\exists\ (P_i)$ P_i PL $\subseteq X$ G-covering X ~~$P_i \subseteq X$ closed~~

Ex. $PL \cup PL \not\Rightarrow PL$

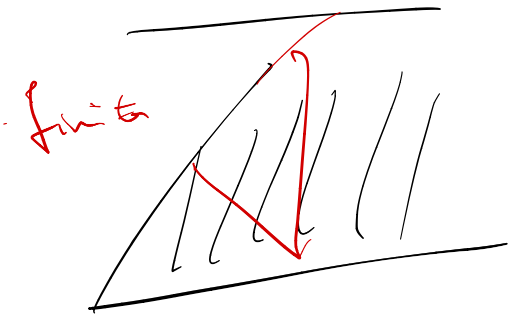
$\underbrace{\{\frac{1}{n}, n \in \mathbb{N}\}}_{PL} \cup \underbrace{\{0\}}_{PL}$ is not PL

Lem. PL $\Rightarrow$ locally closed.

Dim / Tangent space / PL maps $P \xrightarrow{f} \mathbb{R}^n$

$\exists$ locally finite family $(C_i)_i$

s.t. $\begin{cases} \bigcup C_i = P \\ f|_{C_i} \text{ affine.} \end{cases}$



polyhedral
$\longrightarrow$ $\mathbb{R}$

PL
$\longrightarrow$ $\mathbb{R}$

up to ... G-covering

$\to \begin{cases} \text{G-sheaf} \\ \text{G-smooth} \end{cases}$

§1.9 Diff. forms on PL subspace $P \subseteq \mathbb{R}^n$.

$\mathcal{F} \subseteq \mathrm{Fct}(-;\mathbb{R})$ subsheaf ($\mathbb{R}$-vs) $\boxed{\mathcal{F} \text{ is module over } C^\infty(-;\mathbb{R})}$

$$\mathcal{F}^{p,q} := \mathcal{F} \otimes_{\mathbb{R}} A^{p,q}(\mathbb{R}^n)$$ not a G-sheaf.

Idea: $\mathcal{F}^{p,q}_{\underline{P}} : \mathrm{Open}(\underline{P})^{op} \longrightarrow \mathbb{R}\text{-vs}.$

$$U \longmapsto \mathcal{F}^{p,q}_{\underline{P}}(U)$$

$\mathcal{F}^{p,q} / N^{p,q}_{\underline{P}}$ supported on $\overline{\underline{P}}$

$$\rightsquigarrow \mathcal{F}^{p,q}_{\underline{P}} := j^{-1}\left(\mathcal{F}^{p,q} / N^{p,q}_{\underline{P}}\right) \qquad P \overset{j}{\subseteq} \mathbb{R}^n, \quad \subseteq_{\overline{P}} \subseteq$$

$$N^{p,q}_{\underline{P}}$$

Ex. $P = \mathbb{R} \times \{0\} \overset{j}{\subseteq} \mathbb{R}^n$
$\quad\; x \qquad y$

$\omega = dy$. on $\mathbb{R}^n$

$\omega(0) = dy \neq 0$

$j^*\omega = 0 \quad \leftarrow$ more reasonable

$\Rightarrow \omega \in N^{1,0}_{\underline{P}}(\mathbb{R}^n)$

Def. $\omega \in \mathcal{F}^{p,q}(U)$. $\quad U \subseteq \mathbb{R}^n$ open.
$\qquad\qquad\qquad\qquad\quad \cup$
$\qquad\qquad\qquad\qquad\quad \underline{P}$ PL subspace

• $c \subseteq P$ cell $\qquad j_{\langle c\rangle} : \langle c\rangle \hookrightarrow \mathbb{R}^n$

Define: "$\omega|_c = 0$" if $(j^*_{\langle c\rangle}\omega)\big|_c = 0$

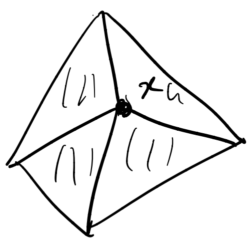
pointwise $= 0$.

• Define: "$\omega|_P = 0$" if "$\omega|_c = 0$" $\forall c \subseteq P$ cell.

Lem $\begin{cases} P \text{ is } G\text{-covered by } (P_i)_i, \quad P_i \subseteq P. \\ \text{"}\omega|_{P_i} = 0\text{"} \end{cases}$

$\Rightarrow$ "$\omega|_P = 0$".

(Idea of) Pf.



For simplicity: $P = C$, $P_i = C_i$ $\qquad \dim C_i \leq \dim C$

$\Rightarrow \exists\, i$ st. $\underline{\dim C_i = \dim C}$

$\Rightarrow \langle C_i \rangle = \langle C \rangle$ $\square$.

$U \subseteq \mathbb{R}^n \supseteq P$

Def. $N^{p,q}_P(U) := \{ \omega \in \mathcal{F}^{p,q}(U) \mid \text{"}\omega|_{P\cap U} = 0\text{"} \}$

By Lem. $\rightsquigarrow N^{p,q}_P$ is a "G-sheaf".

$\Rightarrow$ sheaf.

⚠ $C^\infty(- ; \mathbb{R})$ is not G-sheaf

$f(x) = |x|$

G-smooth = smooth on a G-covering.

$\mathcal{C}^\infty_{P}$

Def. $\mathcal{F}^{p,q}_{P} := j^{-1}\left(\mathcal{F}^{p,q} / \mathcal{N}^{p,q}_{P}\right)$

↑

(p,q)-forms on P of class $\mathcal{F}$.

Lem $\forall\, U \subseteq \mathbb{R}^n$

$$\begin{array}{ccc} \mathcal{F}^{p,q}(U) / \mathcal{N}^{p,q}_{P}(U) & \xrightarrow{\beta} & \mathcal{F}^{p,q}_{P}(P \cap U) \\ & \searrow^{\alpha} \quad \nearrow_{\gamma} & \\ & \left(\mathcal{F}^{p,q} / \mathcal{N}^{p,q}_{P}\right)(U) & \end{array}$$

$P \cap U$ open in P.

① this is $\hookrightarrow$

② this is $\cong$ if $P \cap U \subseteq U$ closed.

Pf. ⓐ Assumption $\mathcal{F}$ module $/ \mathcal{C}^\infty(-;\mathbb{R})$ (fine sheaf)

$\Rightarrow$ $\mathcal{F}$ has partition of unity.

Similarly, $\mathcal{N}^{p,q}_{P}$ has partition of unity.

$$0 \to \mathcal{N}^{p,q}_{P} \to \mathcal{F}^{p,q} \to \mathcal{F}^{p,q} / \mathcal{N}^{p,q}_{P} \to 0$$

$\Gamma(U,-)$

$$0 \to \cdots \to H^1(U, \mathcal{N}^{p,q}_{P}) = 0$$

$\Rightarrow$ α is $\cong$

(b). If $\underline{P} \cap U \overset{j}{\subseteq} U$ closed

$$\mathcal{F}^{p,q}_{\underline{P}} = j^{-1}\Big(\underbrace{\mathcal{F}^{p,q}/\mathcal{N}^{p,q}_{\underline{P}}}_{\text{supported on } \bar{P}\cap U = P\cap U}\Big)$$

$$j_* \mathcal{F}^{p,q}_{\underline{P}} = \mathcal{F}^{p,q}/\mathcal{N}^{p,q}_{\underline{P}} \quad \text{sheaf on } U.$$

$\Gamma(U,-)$

$\Longrightarrow \gamma$ is $\cong$

(a) + (b) $\Longrightarrow$ (2) ✓

For (1): key point: $\mathcal{N}^{p,q}_{\underline{P}}$ is a sheaf.

□

Rmk. (1) $\Longrightarrow$ $\omega \in \mathcal{F}^{p,q}(U)$

$$\omega \in \mathcal{N}^{p,q}_{\underline{P}}(U) \overset{(1)}{\Longleftrightarrow} \boxed{\left\{ \begin{array}{l} \omega|_{\underline{P}\cap U} \in \mathcal{F}^{p,q}_{\underline{P}}(P\cap U) \\ \text{is zero} \end{array} \right.}$$

$\Updownarrow$ def

$$\boxed{\text{“}\omega|_{\underline{P}\cap U} = 0\text{”}}$$

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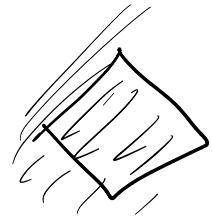
(2) $\Rightarrow$ P PL. $P \overset{\exists}{\subseteq} U$ closed $\quad \mathcal{F}^{p,q}_{\underline{P}}(\underline{P}) \twoheadleftarrow \mathcal{F}^{p,q}(U)$

$\omega \longrightarrow \exists\, \tilde{\omega}$.

$U \subseteq \mathbb{R}^n$ open.

Recall: $\mathcal{F}^{p,q}(U) = \mathcal{F}(U) \otimes_{\mathbb{R}} A^{p,q}(\mathbb{R}^n)$

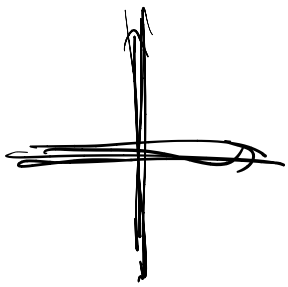
$\Rightarrow$ evaluate at $x \in U$.



Lem/Ex. $P = C$ cell $\subseteq \mathbb{R}^n$

$$\Rightarrow \mathcal{F}^{p,q}_P = \mathcal{F}^{0,0}_P \otimes_{\mathbb{R}} A^{p,q}(\langle \vec{C} \rangle)$$

Non Ex. fails for general P.

 $= P$

$$d'x \wedge d''y = 0 \in A^{1,1}_P(P)$$

but $d'x \wedge d''y \neq 0 \in A^{1,1}(\mathbb{R}^n)$

$=$

$\mathcal{J}, d', d''$ $P \xrightarrow{\varphi} P' \ldots \varphi^*$

$\mathcal{F}^{p,q}_{V_G}$ $V \cong \mathbb{R}^n$

$(\cdot)_G \to G$-topology

(G-covering.)

Def. $V = \mathbb{R}^n$.

V_G = G-topology on V

$\mathrm{Dom}(V)$ $\qquad$ $\{G\text{-coverings}\}$

$\mathrm{Dom}(V) = \{\text{PL subspaces } P \subseteq V\}$

$\{G\text{-coverings}\} = \{G\text{-coverings of PL subspaces by PL subspaces}\}$

Properties:
- $\mathrm{Dom}(V)$ contains opens.
- is closed under $\cap$
- G-coverings of domains are domains.
- open coverings are G-coverings.
- G-coverings can be refined by G-coverings by compact domains.

$F^{p,q}_{P_G}$: $\mathrm{Dom}(P)$

$\rightsquigarrow F^{p,q}_{V_G} : \mathrm{Dom}(V)^{op} \longrightarrow \mathbb{R}\text{-vs}.$

$P \longmapsto F^{p,q}_P(P)$

G-presheaf

sheaf when restricted to $\mathrm{Opens}(P)$.

Rmk. $F^{p,q}_{V_G}$ is separated G-presheaf.

($w|_P = 0$ is a G-local condition)

§ 1.10. positive G-forms

$\underbrace{}$ on P)

section $\in \Gamma(P, \mathcal{F}^{p,q}_{P_G})$

Positivity:

- on a cell $C \subseteq \mathbb{R}^n$.

  $\omega \in \mathcal{F}^{p,q}_C(C)$

  $\rightarrow$ defining positive / w.-positive / str.-positive forms by evaluating ω at any $x \in C$.

  $x \longmapsto \omega(x) \in A^{p,p}(\overrightarrow{\langle C \rangle})$

- $\omega \in \mathcal{F}^{p,q}_{\underline{P}}(\underline{P})$

  $\rightarrow$ define ______ " ______ by restricting to all cells $C \subseteq P$.

Rmk. Enough to check for C_i in some G-covering $(C_i)_i$ of P.

Prop $P \subseteq \mathbb{R}^n$ PL subspace

1). sym. sm. (p,p)-form $\alpha \in A^{p,p}_{\underline{P}}(P)$

$\Rightarrow \alpha = \alpha_1 - \alpha_2$

sym. sm. positive (p,p)-form.

2) P compact, $u \in A_P(P)$

$\Rightarrow u = u_1 - u_2$. s.t. $d'd'' u_i \geq 0$.

<u>Locally</u> $^{(1)+(2)}$: $\lambda = d'd'' \|x\|^2$

<u>Idea</u>: $\alpha = \underbrace{(\alpha + c \cdot \lambda^P)}_{\text{positive if}} - \underbrace{c \cdot \lambda^P}_{\text{positive}}$

$c >> 0$.

($\lambda^P \in$ interior of cone of positive forms)

<u>Globally</u>: (1)

partition of unity:

$\alpha = \alpha + \psi \cdot \lambda^P$

($\psi \geq c$ locally)

$\boxed{\mathrm{Conv}_P}$

when P convex $\Rightarrow$ usual convexity.

1.19.11 $\qquad$ $G_{\mathbb{R}}$-smooth $\rightsquigarrow$ G-smooth

Prop $\quad P \subseteq \mathbb{R}^n$. compact cell (bounded)

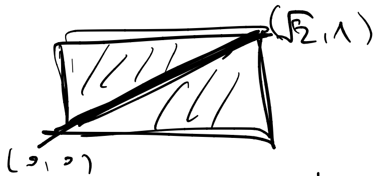
1) $\forall u \in \Gamma(\underbrace{P, A_{P_G}}_{\text{G-smooth functions on } P}) \Rightarrow u = u_1 - u_2$

u_1, u_2 G-smooth & convex on P.

2) $\forall u$ PL-function $\Rightarrow u = u_1 - u_2$

u_1, u_2 PL & convex.

Rmk. PL $\Rightarrow$ G-smooth
$\quad\;\; \|$
G-affine



$f(x,y) = |y - \sqrt{2}x|$
$G_{\mathbb{R}}$-smooth
not G-smooth
($G_{\mathbb{Q}}$-smooth)

Idea:

1) $\overset{\text{G-sm}}{u} = \left(u + c \cdot \|x\|^2\right) - \|x\|^2$

2) $\overset{\text{PL}}{u} = (u + |f_1| + |f_2| + \dots) - (|f_1| + |f_2| + \dots)$

Proof $\quad$ G-covering $= (C_i)_i = \mathscr{C}$

we may assume u convex on each cell C_i.

we will deal with 1) + 2) simultaneously.

we assume $\langle P \rangle = \mathbb{R}^n$.

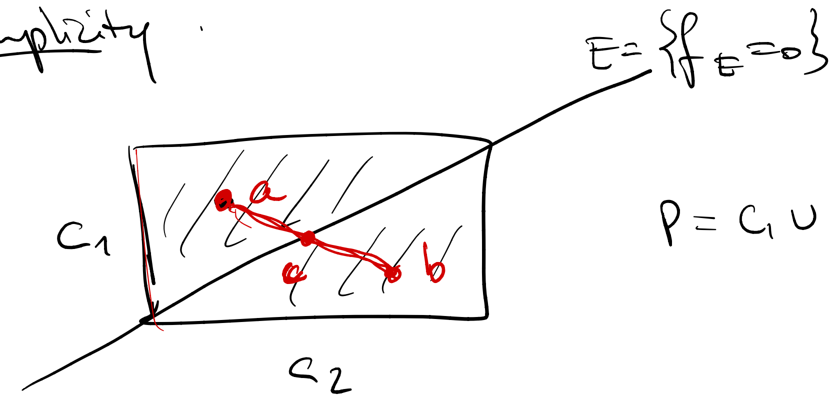
$$\mathcal{E} = \left\{ \text{affine hyperplanes } \langle C \rangle \;\middle|\; \begin{array}{l} C \in \mathscr{C} \\ \dim C = n-1 \end{array} \right\}$$

$\forall E \in \mathcal{E}$, fix f_E affine form. " $\{f_E = 0\}$ "

$$u = \underbrace{\left(u + \sum_{E \in \mathcal{E}} \lambda_E |f_E|\right)}_{\upsilon_\lambda} - \left(\sum_{E \in \mathcal{E}} \lambda_E |f_E|\right)$$

<u>Hope</u>: $\lambda_E >> 0 \Rightarrow v_\lambda$ is convex.

<u>For simplicity</u>:



$P = C_1 \cup C_2$

<u>Convexity</u>:

$\forall\, a, b \in P$

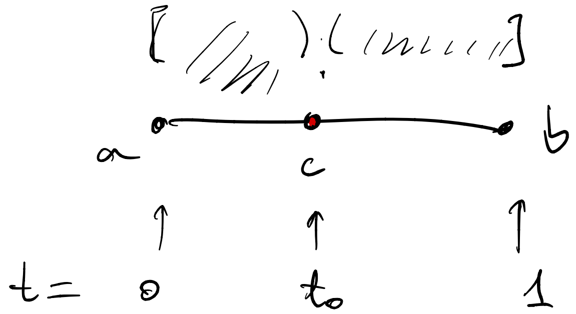
$f(t) := v_\lambda((1-t)a + tb)$ is <u>convex</u> for $\lambda_E >> 0$?

for $t \in [0,1]$.

(1) $a, b \in C_1$

or $a, b \in C_2$.

(2) $a \in \mathring{C}_1$. $b \in \mathring{C}_2$



<u>check</u>: $f'(t_0^+) - f'(t_0^-) \geq 0$?

for $\lambda_E >> 0$

$$= \langle \nabla u_{C_1}(c) - \nabla u_{C_2}(c) + 2\,\varepsilon_E \lambda_E \nabla f_E , b-a \rangle$$

$\varepsilon_E = \pm 1$, depending on

$\langle \nabla f_E , b-a \rangle$ > 0
< 0

$$\Rightarrow \langle \varepsilon_E \nabla f_E , b-a \rangle = |\langle \nabla f_E , b-a \rangle| \neq 0$$

$\Rightarrow \lambda_E \gg 0$ get ≥ 0

But P compact $\Rightarrow$ can choose λ_E uniformly for a, b
$\in \mathring{C}_1$, $\in \mathring{C}_2$.

$\square$