CLD

§10.1 – §10.3 + ε

29 Nov 2025

§10.1 - Graded $\overset{\text{comm.}}{\text{alg}}$

§10.2 - ——— schemes

§10.3 - Strict graded alg/schemes.

Def. $(\mathbb{R}_+^\times-)$ graded ring $=$ ring $A = \bigoplus_{r>0} A^r = \bigoplus_{r>0} A^{[r]}$
$(\mathbb{R}_{>0}, \times)$

degree / grading

st. A^r ab. grp.
$A^r \times A^{r'} \longrightarrow A^{r \cdot r'}$ $\Big\} \Rightarrow A^{[1]}$ classical ring

Morphism: $A \to B$ ring homo.
st. $A^r \to B^r$

$\rightsquigarrow$ GrRing.
$\cup$
$\mathrm{GrRing}^{\mathrm{triv}} = \{ A = A^{[1]} \} \xleftarrow{\cong}$ Ring

Def. $A \in$ GrRing. $\rightsquigarrow$ $M \in$ Gr-A-Mod.
$M = \bigoplus_{r>0} M^r$ $\quad A^r \times M^{r'} \to M^{rr'}$

Morphism: $M \to N$. A-mod morphism
$\Updownarrow$ $\quad M^r \to N^r$ (no shifts)
deg=0 morphism

$\rightsquigarrow \mathrm{Hom}_{A,0}(M,N)$ only has $A^{[1]}$-module

$\boxed{\text{internal Hom}}$ (allow shifts) $\qquad \boxed{\text{graded A-mod}}$

$\underline{\mathrm{Hom}}_A(M,N) = \bigoplus_{\rho>0} \underbrace{\mathrm{Hom}_{A,0}(M, N[\rho^{-1}])}_{\deg = \rho}$

$N \qquad \cdots \oplus N^{1/2} \oplus \cdots \oplus N^1 \oplus \cdots \oplus N^2 \oplus \cdots$

$N[p^{-1}] : \quad \cdots \oplus N^{\frac{1}{2}p} \oplus \cdots \oplus N^{p} \oplus \cdots \oplus N^{2p} \oplus \cdots$

shift to the left by p (multiplication on deg by p)

Rmk. [CLD] 10.1.4 last paragraph on matrices....

$\longrightarrow$ should also allow degree shifts.

Def. Free A-mod. $\bigoplus_{i \in I} A[r_i]$ for some $r_i > 0$

$\rightsquigarrow$ matrix.

Def. Graded field $= K$ graded ring where $\forall a \in K^{\text{hom}} \setminus \{0\}$ $\exists a^{-1}$ $\quad 1 \in A^{[1]}$

$(a \in K^r \setminus \{0\} \Rightarrow \exists a^{-1} \in K^{r^{-1}} \setminus \{0\})$

k (classical) field

Ex. $k[t, t^{-1}]$ graded field. $= \bigoplus_{i \in \mathbb{Z}} \boxed{k} t^i$

$|t| = 2$.

$|t^{-1}| = \frac{1}{2}$

$k[t, t^{-1}]^{[1]} = k$

Notation: $A \in \mathrm{GrRing}$. $\underset{\neq 0}{a \in A^r} \Rightarrow \|a\| = \deg(a) = r$.

$a = 0 \Rightarrow \|a\| = 0$.

K Gr field $\Rightarrow \|\cdot\| = |\cdot|$

$K \Rightarrow K^\times = K^{\text{hom}, \times}$

$|K^\times| = \{$ degrees of $K^\times \}$

Cor. $\forall$ Graded K-Mod is free $\left(= \bigoplus_{i \in I} K[r_i] \right)$

Polynomial ~~rings~~ graded A-alg. $\quad A \in \mathrm{GrRing}$.

$r = (r_i)_{i \in I} \quad r_i > 0.$ multiindex.

$$A[T/r] := \underset{(T_i)_{i\in I}}{A[T]} \quad \text{with } |T_i| = r_i.$$

univ. property

$$\begin{array}{ccc} \mathrm{Hom}_{A\text{-Ring}}(A[T/r], B) & \xrightarrow{\cong} & \mathrm{Map}(I, B) \\ & & i \mapsto \deg r_i \\ f & \longmapsto & (f(T_i))_{i \in I} \end{array}$$

(k classical field)

Ex. $k[t, t^{-1}] \leftarrow |t| = 2, \quad |t^{-1}| = \frac{1}{2}$

$$\| $$

$$k[t/2, u/\tfrac{1}{2}] / (\underbrace{ut}_{\deg 1} - 1) = k[t/2, 2u]/(ut-1)$$

Def. $A\left(\frac{T}{r}\right) := \mathrm{Frac}\left(A\left[\frac{T}{r}\right]\right)$

invert all homogeneous elements

$$k[t/2] \rightsquigarrow k(t/2) = k[t, t^{-1}].$$

- Field theory

Algebraic independence of $(a_i)_{i \in I}$. $\quad / K \in \mathrm{GrField}$

$$K\left(\frac{T}{r}\right) \hookrightarrow L$$
$$T_i \longmapsto a_i$$
$$|a_i| = r_i$$

$\rightsquigarrow \deg tr(L/K)$

$[L : K] := \dim_K L$

Formulas: ① $[L:K] = [L^1 : K^1] \cdot [|L^\times| : |K^\times|]$ $\quad \left(= \# \left(|L^\times| / |K^\times| \right)\right)$

② $\deg tr(L/K) = \deg tr(L^1/K^1) + \dim_{\mathbb{Q}}\left(|L^\times| / |K^\times| \otimes_{\mathbb{Z}} \mathbb{Q} \right)$

$|K^\times| = \{ r \mid K^r \neq 0 \}$

pf. ① $K = \bigoplus_{r \in |K^\times|} K^r \hookrightarrow \overbrace{K \otimes_{K^1} L^1}^{M} = \bigoplus_{r \in |K^\times|} L^r \hookrightarrow L = \bigoplus_{r \in |L^\times|} L^r$

L^1/K^1 "$\updownarrow$" residue field extension unramified

$\# (|L^\times| / |K^\times|)$ "$\updownarrow$" totally ramified

② $K \hookrightarrow M \hookrightarrow L$

$\operatorname{degtr}(L^1/K^1)$

$|L^\times| / |K^\times| \otimes_{\mathbb{Z}} \mathbb{Q} = \underbrace{\langle r_1, r_2, \cdots, r_n \rangle}_{\exists}$

$\forall \lambda \in L$, $|\lambda| \in |L^\times|$

$\Rightarrow \exists N \in \mathbb{N}_{>0}$.

$|\lambda^N| \in r_1^{\alpha_1} \cdot r_2^{\alpha_2} \cdots r_n^{\alpha_n} \cdot |K^\times|$

let $b_i \in L^\times$ st. $|b_i| = r_i$

$\Rightarrow |\lambda^N| \in |b^\alpha| \cdot |K^\times|$

$\Rightarrow \lambda^N = \underset{M\left(\frac{b_i}{r_i}\right)}{b^\alpha} \cdot \underbrace{\lambda^N / b^\alpha}_{\in M}$

$\Rightarrow \lambda$ alg over $M\left(\frac{b_i}{r_i}\right)$

$\operatorname{degtr}(-/M) = n$.

$\Rightarrow \operatorname{degtr}(L/M) \leq n$ "$=$"

Ex. k complete DVF

$k \supset k^\circ = \{ \|\cdot\| \leq 1 \} \twoheadrightarrow \tilde{k}$

$k^\circ \supset k^{\circ\circ}$ $\|\cdot\| < 1$

(~~§10.8~~)

§10.8. graded reduction

$\tilde{k}^{(gr)} = \bigoplus_{r > 0} \{ \|\cdot\| \leq r \} / \{ \|\cdot\| < r \}$

$\tilde{k}^{gr}$ graded field.

k'/k ext. of complete DVF

$\rightsquigarrow$ $\underset{L}{\widetilde{k'}^{gr}} / \underset{K}{\tilde{k}^{gr}}$ $\tilde{k}^{gr} \underset{unr}{\subset} M \underset{\text{tot. ram.}}{\subset} \widetilde{k'}^{gr}$

$\Rightarrow$ $\underbrace{[M : \tilde{k}^{gr}]}_{f} \cdot \underbrace{[\widetilde{k'}^{gr} : M]}_{e} = [\widetilde{k'}^{gr} : \tilde{k}^{gr}] \leq [k' : k]$

if not complete

$\boxed{= \text{ if complete}}$

(degtr .... similarly)

Ex. $k = \mathbb{Q}_p$, $|p| = \frac{1}{2}$ $\qquad \tilde{k}^{gr} = \mathbb{F}_p[t, t^{-1}]$, $|t| = \frac{1}{2}$.

$k = \mathbb{F}_p((t))$ $\qquad \tilde{k}^{gr} = \mathbb{F}_p[t, t^{-1}]$

~~similarly for GrRing~~

~~$\mathbb{Q}_p \langle T_1/r_1, \ldots, T_n/r_n \rangle^{gr}$~~ $= \mathbb{F}_p\left[\frac{t_1}{r_1}, \ldots, \frac{t_n}{r_n}, \frac{t}{\frac{1}{2}}\right]$ $\leftrightarrow p$

- (graded) Valuation ring : K graded field.

$v: K^\times \to K^\times/\mathcal{O}^\times = G$

v valuation $:=$ $v : K^{\times, hom} \longrightarrow (G, \times)$ ~~ordered~~ ab. grp.

$v(\lambda\mu) = v(\lambda) \cdot v(\mu)$

$v(\lambda + \mu) \leq \max(v(\lambda), v(\mu))$

bij.

$\mathcal{O}$ $\qquad \mathcal{O}_v$ valuation ring : $\mathcal{O} \subseteq K$ graded subring st. $\forall \lambda \in K^\times \Rightarrow \lambda \in \mathcal{O} / \lambda^{-1} \in \mathcal{O}$.

$\mathcal{O}_v = \bigoplus_{r>0} \{\lambda \in K^r \mid v(\lambda) \leq 1\}$

§10.2 Graded comm. alg / schemes.

$A \in \mathrm{GrRing}$. Graded ideals of A. $\left(\overset{\text{Gr-}}{A\text{-mod}} \subseteq A \right)$

- the classical nilradical of A is a graded ideal.

$$(x_{r_1} \oplus x_{r_2} \oplus \cdots \oplus x_{r_n})^n = 0 \Rightarrow x_{r_1} \text{ nilpotent}$$
$$\Rightarrow x_{r_2} \text{ nilpotent}$$
$$\Rightarrow \cdots x_{r_i} \text{ nilpotent}$$

- A integral ring (classically) $\Rightarrow$ A integral graded ring

$x \cdot y \in A^{hom} \setminus \{0\} \Rightarrow x \cdot y \neq 0.$

- graded Prime ideals of A $\rightsquigarrow$ $\mathrm{Spec}(A)$, $D(f)$, $f \in A^{hom}$

$\mathrm{Spec}^{\text{classical}}(k[t, t^{-1}])$ 1-dim.
$\mathrm{Spec}(k[t, t^{-1}])$ 0-dim. point.

- Noeth. / Art., graded

- $\mathrm{Spec}(A) \overset{\text{glue}}{\rightsquigarrow}$ graded schemes

$\Updownarrow$

$(|X|, \mathcal{O}_X)$ locally $\cong \mathrm{Spec}\, A$. $\qquad \mathcal{O}_X = \bigoplus_{r \geq 0} \mathcal{O}_X^{[r]}$

- Proper $\Leftrightarrow$ finite type + sep. + universally closed
$\Leftrightarrow$ valuative criterion

$$\begin{array}{c} K \\ \cup \\ \mathcal{O} \end{array} \qquad \begin{tikzcd} \mathrm{Spec}(K) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(\mathcal{O}) \arrow[r] \arrow[ur, dashed, "\exists !"] & Y \end{tikzcd}$$

Facts: ① $\forall A$. local ring, integral. $K = \mathrm{Frac}(A)$

$\Rightarrow A \underset{\text{dominant}}{\subset} \exists \underset{\text{val. ring}}{\mathcal{O}} \subset K$

$n \cap A = m$. $\rightsquigarrow n$

② $Y \xrightarrow{\text{q.c.}} X$ is closed $\Leftarrow\!\Rightarrow$ specialising?

$$\begin{array}{ccc} \forall y & \rightsquigarrow & \exists y' \\ f \downarrow & & \downarrow \\ x & \rightsquigarrow & \forall x' \end{array}$$

$P_x \ \mathrm{sp.} \subseteq P_{x'}$

③ going up. integral ring map $A \to B$

$\Rightarrow \mathrm{Spec}(B) \to \mathrm{Spec}(A)$

closed / specialising.

- $\mathcal{O}_X$-mod, $VB(X) \rightsquigarrow$ locally free $\left(\cong \bigoplus_{i \in I} \mathcal{O}_X[r_i] \right)$ $r_i > 0$.

$\mathrm{Hom}_{\mathcal{O}_X}(E, F)$ — no shifts $\mathcal{O}_X^{[\cdot]}$-mod

$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(E, F)$ — with shifts $\mathcal{O}_X$-mod.

$\mathcal{O}_P(X)^{op} \longrightarrow \underline{\mathrm{Gr}}\ \mathrm{AbGrp}$.

$\mathcal{O}_X^{[r]} \times \bigoplus_{r>0} F^{r^*} \longrightarrow \ldots$

- $\mathbb{A}^r_A$ $\quad r = (r_1, \ldots, r_n)$

$\mathbb{P}^r_A = \mathrm{Proj}\left(A\left[\frac{T_0}{r_0}, \ldots, \frac{T_n}{r_n}\right]\right)$ $\quad r = (r_0, \ldots, r_n)$

$\cup$

$(\mathbb{R}^\times_+)$-graded prime ideals.

that are homogeneous ~~in~~ in monomials.

generated by: $\sum a_I T^I \ (\in p)$

$|a_I| \cdot r^I = \text{same}$ $\quad (\mathbb{R}^\times_+ - \text{grading})$

$|I| = \text{same}$ $\quad (\leftrightarrow \mathrm{Proj.})$

$\mathcal{O}(1)$ on $\mathbb{P}^r_A$. $\leftrightarrow$ homogeneous / monomials A-mod $\subseteq A[T_0, \ldots, T_n]$

gen. $(T_0, \ldots, T_n)$

$$\mathcal{O}(1)\Big|_{T_j \neq 0} \cong \underbrace{A\left[\frac{T_i}{r_i / r_j}\right]_{i \neq j}}\cdot T_j = \underbrace{\mathcal{O}_{U_j}}[\underset{\substack{\shortparallel \\ |T_j|}}{r_j}]$$

$$\mathcal{O}_{u_i \cap u_j}[r_i] \overset{\simeq}{\cong} \mathcal{O}_{u_i \cap u_j}[r_j]$$

$$|c_{ij}| = \frac{r_i}{r_j} \qquad |c_{ij}^{-1}| = \frac{r_j}{r_i}$$

$$k[t, t^{-1}][2] \cong k[t, t^{-1}]$$
$|t|=2$

$\rightsquigarrow$ very ~~dumple~~. ~~ample~~ ample. (using $\mathcal{O}(1)$)

§10.3. Strict

$\boxed{K \in \mathrm{GrField}}$

$A \in \mathrm{GrRing}/K \qquad A^{[1]} = A^1 / K^1$

$\boxed{\text{strict}}$

$K^1\text{-alg} \rightsquigarrow \mathrm{St}\text{-}K\text{-alg} \hookrightarrow K\text{-alg}$

$A^1 \longmapsto \underbrace{A^1 \otimes_{K^1} K}_{\text{strict}} \longrightarrow A \qquad$ Gr Ring map.

Def. A is $\boxed{\text{strict}/K}$ if $A^1 \otimes_{K^1} K \xrightarrow{\cong} A$

$$\Rightarrow \underbrace{A^r = A^1 \otimes_{K^1}}_{K^{r-1}} \underbrace{K^r}_{\boxed{\text{invertible}}}$$

$\boxed{\text{Scaling Trick}}$ $|a_r| = r \Rightarrow \exists\, b_r \in K^r \setminus \{0\}$
$\in |K^\times|$

$$a_r = \underset{\in A^1}{(b_r^{-1} a_r)} \otimes \underset{\in K^r}{b_r}$$

Lem 10.3.1: $A^1 \otimes \to A^1 \otimes_{K^1} K \xrightarrow{\cong} A \in \mathrm{St}\text{-}K\text{-alg}.$

$$p^1 \longleftarrow p$$

$$\mathrm{Spec}(A^1) \longleftarrow \mathrm{Spec}(A)$$

(1) this $\cong$ on top. spaces.
(preserving closed subspace)

(2) $\mathrm{Spec}(A^1)$ $\overset{A^1}{\Leftarrow}$ is integral / reduced / local / noeth / artinian

valuation ring

$\iff$ $\mathrm{Spec}(A)$ $\overset{\Leftarrow}{\scriptstyle A}$ is ______ " ______

— " ______

(3) closed subscheme $\subseteq \mathrm{Spec}(A^1)$

$\Updownarrow$

closed graded subscheme $\subseteq \mathrm{Spec}(A)$

(4) M / A^1 VB $\iff$ $M \otimes_{A^1} A$ $/A$ VB

(5) res. field. $x^{[1]} \longleftrightarrow x$

$\kappa(x^{[1]})$ $\kappa(x)$

$$\Rightarrow \kappa(x) = \underbrace{\kappa(x^{[1]})}_{\substack{\parallel \\ \kappa(x)^1}} \otimes_{K^1} K$$

(6) multiplicity / length match.

Lem $\Rightarrow$ $X = \mathrm{Spec}\, A = (|\mathrm{Spec}\, A^1|, \mathcal{O}_X)$

$X^1 = \mathrm{Spec}(A^1)$. $\mathcal{O}_X^{[1]} = \mathcal{O}_{X^1}$

- Strict: ~~[illegible]~~ $\Rightarrow$ $|A^\times| = |K^\times|$

Ex. $K[\frac{T}{r}]$ strict if $r \in |K^\times|$ $T \leftrightarrow a \cdot a^{-1}T$

$\cong K[\frac{T}{1}] = K[T]$, $|T| = 1$ $|a| = r$ $|a^{-1}T| = 1$

Ex $A/_{k}$ strict Tate alg. $\rightsquigarrow$ $\tilde{A}$ = graded reduction is strict $\tilde{k}$-alg

(Ex. 10.4.3) let $r > 0$. $\{r$ non-torsion modulo $|k^\times|\}$

Ex. $K(T/r) = K[\frac{T}{r}, rS] / (TS-1)$ field, not strict $/K$.

Let $F/_K$ field ext, st. $|F^\times| \ni r$,

$\Downarrow$

$K(T/r) \otimes_K F$ is strict $/F$, $\cong F[\frac{T}{r}, rS]/(TS-1) \overset{r \in |F^\times|}{\cong} F[T, S]/(TS-1)$

(graded field) $\otimes_K F$ becomes $\mathbb{G}_{m,F} = \mathbb{G}_{m,F^1} \otimes_{F^1} F$ $\underset{\dim = 1}{}$ $\overset{\parallel}{F[T^{\pm 1}]}$

In general, Splitting field:

L/k splits X/k if X_L/L is strict.

Ex. X/k finite type. $(E_i)_{\text{finitely many}}$ $VB(X)$

$\rightsquigarrow \exists$ finitely many $p_1, \ldots, p_n > 0$

s.t. $\underbrace{K(T/p)}_{\text{transcendental}}$ splits X & E_i.

(descend $\underline{\text{irreducible components}}$)

$$X_p := X \otimes_k K(T/p).$$

Lem.
- X/k
- X_p strict $/K(T/p)$

transcendence for $\Rightarrow$. X_p strict. classical

$\Rightarrow$ (a). X irred. $\Leftrightarrow$ X_p irred $\Leftrightarrow$ $(X_p)^1$ irred
red. red red

(b). $X_i \subseteq X$ irred comp. $\rightsquigarrow$ $\operatorname{mult}_{X_i}(X)$
$\qquad = \operatorname{mult}_{(X_{i,p})^1} (X_p)^1$

Lem. L/k splits X/k $\Rightarrow$ M/L splits X

& $(X_L)^1 \otimes_{L^1} M^1 = (X_M)^1$

& similarly for $E \in VB(X)$.

Ex. $X \overset{i}{\hookrightarrow} \mathbb{P}^r_K$ immersion, E line bundle over X

$D \qquad \mathcal{O}(1)$

$\exists\, m \gg 0$. s.t. $D^{\otimes m}$, $D^{\otimes m} \otimes E$ generated by global sections.

pf. $L := K(T/p)$ splits $\mathbb{P}^r_K$, & $\mathcal{O}(1)$, E

$\Rightarrow$ splits X, D.

$(X_p)^1$, $(D_p)^1$ very ample since D_p is very ample

($\Leftarrow D \ldots$)

$\Rightarrow \exists\, m \gg 0$. $(D_p)^{1,\otimes m}$ $(D_p)^{1,\otimes m} \otimes (E_p)^1$ ample ①

$\Rightarrow$ generated by global sections. ②

$\Updownarrow$ strict

$D_p^{\otimes m}$, $D_p^{\otimes m} \otimes E_p$. ① $\boxed{②}$

$\Downarrow \qquad\qquad$ for any field extension $\boxed{\Downarrow}$

$D^{\otimes m}$, $D^{\otimes m} \otimes E \qquad \boxed{②}$

✓

$\mathrm{Spec}(A)$ proper$/K$ $\Rightarrow$ $\mathrm{Spec}(A^1)$ proper$/K^1$

$\not\Leftarrow$?

ERRATUM of notes (see red words for conditions I forgot):

(Ex. 10.4.3) let $r > 0$. $\boxed{r \text{ non-torsion modulo } |k^\times|}$

Ex. $K(T/r) = K[\frac{T}{r}, rS]/(TS-1)$ field, not strict$/K$.

Let F/K field ext, s.t. $|F^\times| \ni r$,

$\Downarrow$

$K(T/r) \otimes_K F$ is strict$/F$ since $\cong F[\frac{T}{r}, rS]/(TS-1) \cong F[\tau, \sigma]/(\tau\sigma - 1) = F[\tau^{\pm 1}]$ ($r \in |F^\times|$) torus

Summary: for r non-torsion mod $|k^\times|$, F/k s.t. $|F^\times| \ni r$:

- graded field $K(T/r)/K$, Spec has $\underset{\text{Krull}}{\dim} = 0$, not strict$/K$ is finite type K-alg.
- but the finite type F-alg: $K(T/r) \otimes_K F$, Spec $= \mathbb{G}_{m,F} = \mathbb{G}_{m,F_1} \otimes_{F_1} F$ has $\underset{\text{Krull}}{\dim} = 1$; $\dim = 1$ strict$/F$

$\Longrightarrow$ Krull dimension is not well-behaved under base change.