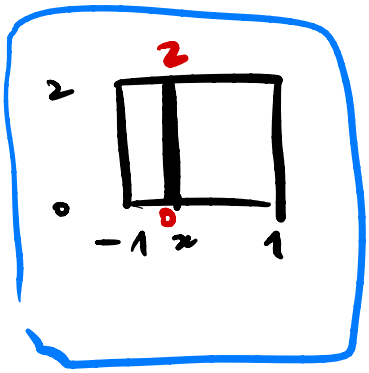
TD analyse 12/11/2026

Exercice 7 (intégration double)

(1) – (3)



(1) $D = \{(x,y) : -1 \le x \le 1 \ , \ 0 \le y \le 2\}$

$$\iint_D (x+y)^2 \, dx\,dy$$

$$= \int_{-1}^{1} \left(\int_0^2 (x+y)^2 \, dy \right) dx$$

$$= \int_{-1}^{1} \left[\frac{1}{3}(x+y)^3 \right]_0^2 dx$$

$$= \int_{-1}^{1} \frac{1}{3}\left((x+2)^3 - x^3\right) dx$$

$$= \frac{1}{3} \int_{-1}^{1} \left((x+2)^3 - x^3\right) dx$$

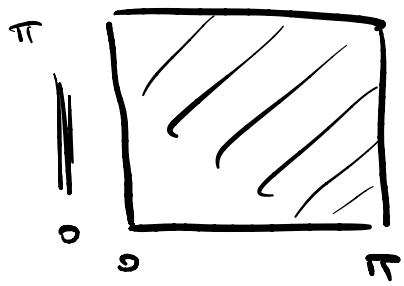
$$= \frac{1}{3} \left[\frac{1}{4}(x+2)^4 - \frac{1}{4}x^4 \right]_{-1}^{1}$$

$$= \frac{1}{3} \cdot \frac{1}{4} \left[\left(3^4 - 1^4\right) - \left(1^4 - (-1)^4\right) \right]$$

$81 - 1 = 80$

$$= \frac{20}{3}$$

(2) $D = \{ 0 \le x \le \pi, \ 0 \le y \le \pi \}$

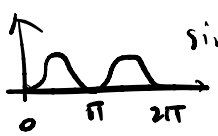


$$\iint_D \sin^2(x-2y) \, dx\,dy$$

$$= \int_0^{\pi} \left(\int_0^{\pi} \sin^2(x-2y) \, dy \right) dx$$

$$\int_0^{\pi} \frac{1 - \cos(2x-4y)}{2} \, dy$$

$$\int_0^{\pi} \sin^2 x \, dx = \frac{1}{2} \times \pi$$



$2 \cdot \frac{\pi}{2}$

$$\int_0^{\pi} \sin^2(2x) \, dx = \frac{1}{2} \times 2x \frac{\pi}{2} = \frac{\pi}{2}$$

période $\frac{\pi}{2}$

$$\int_0^{\frac{\pi}{\alpha}} \sin^2(\alpha x)\,dx = \frac{1}{2} \times \frac{\pi}{\alpha} \qquad \alpha > 0$$

$$\frac{1}{2}\pi - \frac{1}{2}\int_0^{\pi} \cos(2x-4y)\,dy$$

période $\frac{2\pi}{4}$ (par rapport à y)

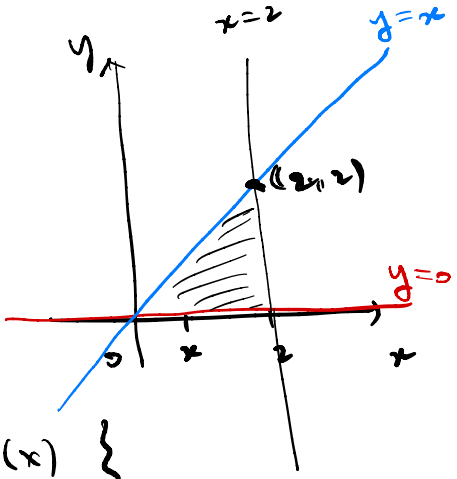
$$= 0$$

$$\left(-\frac{1}{2}\cdot\left(-\frac{1}{4}\right)\sin(2x-4y)\Big|_0^{\pi} = 0 \right)$$

$$\frac{1}{2}\pi$$

$$= \int_0^{\pi} \frac{1}{2}\pi\,dx = \frac{1}{2}\pi^2$$

(3) D = l'intérieur de $\begin{matrix} x = 2 \\ y = 0 \\ y = x \end{matrix}$ =



$$= \{ 0 \le x \le 2,\ a(x) \le y \le d(x) \}$$

$a(x) = 0$, $d(x) = x$

$$= \{ 0 \le x \le 2,\ 0 \le y \le x \}$$

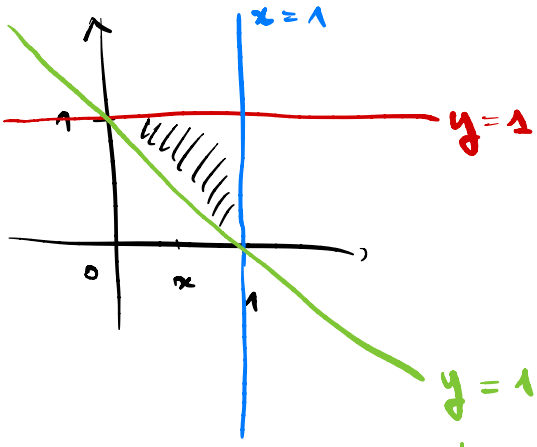
$$\iint_D (x+y)^2\,dx\,dy = \int_0^2 \left(\int_0^x (x+y)^2\,dy \right) dx$$

$$\int_0^x (x+y)^2\,dy = \left[\frac{1}{3}(x+y)^3 \right]_0^x = \frac{1}{3}\left((2x)^3 - x^3\right) = \frac{7}{3}x^3$$

$(2x)^3 = 8x^3$

$$\Rightarrow \iint_D (x+y)^2\,dx\,dy = \int_0^2 \frac{7}{3}x^3\,dx = \frac{7}{3}\cdot\left[\frac{1}{4}x^4\right]_0^2 = \frac{7}{3}\cdot 4 = \frac{28}{3}$$

(4) $D = \begin{cases} x \le 1 \\ y \le 1 \\ y \ge 1-x \end{cases}$



$0 \le x \le 1$, pour chaque x :

$1-x \le y \le 1$

$(1-x \le 1$ pour $0 \le x \le 1)$ ✓

(évaluer
$x=0 \rightsquigarrow 1$
$x=1 \rightsquigarrow 0$)

$\Rightarrow D = \{ 0 \le x \le 1 \,.\, 1-x \le y \le 1 \}$

$$\iint_D (x+y)^2 \, dx\,dy = \int_0^1 \left(\int_{1-x}^{1} (x+y)^2 \, dy \right) dx$$

$$= \int_0^1 \left[\frac{1}{3}(x+y)^3 \right]_{y=1-x}^{1} dx$$

$$= \frac{1}{3} \int_0^1 \left[(x+1)^3 - \underbrace{(x+1-x)^3}_{1} \right] dx$$

$$= \frac{1}{3} \int_0^1 \left((x+1)^3 - 1 \right) dx$$

$$= \frac{1}{3} \left(\left[\frac{1}{4}(x+1)^4 \right]_0^1 - 1 \times 1 \right)$$

$$= \frac{1}{3} \left(\left(\frac{1}{4} \times 2^4 - \frac{1}{4} \times 1^4 \right) - 1 \right)$$

$4 - \frac{1}{4} - 1 = 3 - \frac{1}{4} = \frac{11}{4}$

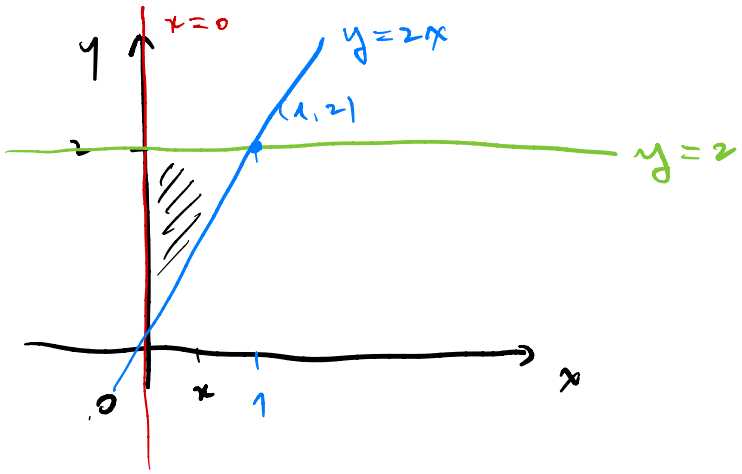
$$= \frac{11}{12}$$

(5) $x=0$

$y=2$

$y=2x$

2	1
0	0



$$D=\{0\le x\le 1,\quad 2x\le y\le 2\}$$

$$\Rightarrow \iint_D (x+y)^2\,dx\,dy = \int_0^1\left(\int_{2x}^{2}(x+y)^2\,dy\right)dx$$

$$=\int_0^1\left[\frac{1}{3}(x+y)^3\right]_{2x}^{2}dx$$

$$=\frac{1}{3}\int_0^1\left((x+2)^3-(3x)^3\right)dx$$

$(x+2)^3-27x^3$

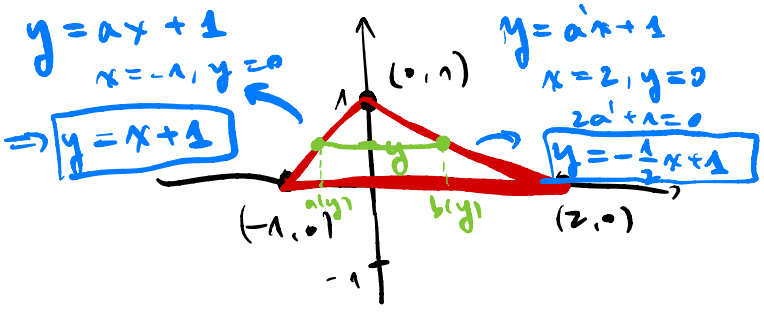
$$=\frac{1}{3}\left[\frac{1}{4}(x+2)^4-\frac{27}{4}x^4\right]\Big|_0^1$$

$$=\frac{1}{12}\left[(x+2)^4-27x^4\right]\Big|_0^1$$

$$=\frac{1}{12}\left[(3^4-2^4)-27\right]=\frac{38}{12}=\frac{19}{6}$$

$81-16-27$

(6) $(-1,0)$, $(0,1)$, $(2,0)$



$-1\le x\le 2$,

si $-1\le x\le 0$, $0\le y\le x+1$

si $0\le x\le 2$, $0\le y\le -\frac{1}{2}x+1$

fixer d'abord y. écrire inconditionnellement

$$a(y) \le x \le b(y)$$

$a(y)$ vérifie : $y = a(y) + 1 \Rightarrow a(y) = y - 1$

$b(y)$ vérifie $y = -\frac{1}{2} b(y) + 1 \Rightarrow b(y) = 2 - 2y$

y varie dans $[0,1]$ d'après le graphe.

$$\Rightarrow \boxed{D = \{ 0 \le y \le 1, \quad y - 1 \le x \le 2 - 2y \}}$$

$$\iint_D (x+y)^2 \, dx\, dy = \int_0^1 \left(\int_{\underbrace{y-1}_{=a(y)}}^{\underbrace{2-2y}_{=b(y)}} (x+y)^2 \, dx \right) dy$$

$$= \int_0^1 \left[\frac{1}{3} (x+y)^3 \right]_{x=y-1}^{2-2y} dy$$

$$= \frac{1}{3} \int_0^1 \left((2-y)^3 - (2y-1)^3 \right) dy$$

$$= \frac{1}{3} \left(\left[-\frac{1}{4} (2-y)^4 \right]_0^1 - \left[\frac{1}{2} \times \frac{1}{4} (2y-1)^4 \right]_0^1 \right)$$

$$= \frac{1}{3} \times \frac{1}{4} \times \left(- \left[(2-y)^4 \right]_0^1 - \frac{1}{2} \left[(2y-1)^4 \right]_0^1 \right)$$

$$= \frac{1}{12} \times \left(- \left(1^4 - 2^4 \right) - \frac{1}{2} (1 - 1) \right)$$

-15

$$= \frac{1}{12} \times 15 = \frac{5}{4}$$

Ex 7. (7) – (12) prochaine fois.

$D = \{ a \le x \le b, \ c(x) \le y \le d(x) \}$

ou inverser les rôles de x, y.

Ex 8 (coordonnées polaires)

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad r \ge 0, \quad \theta \in [0, 2\pi] \ ([0, 2\pi[)$$

$$dx\,dy = \boxed{r}\, dr\,d\theta$$

↑ jacobien $= \det \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

(1) $\iint x^2 \, dx\,dy = \iint_{\{0 \le r \le 2,\ 0 \le \theta \le 2\pi\}} r^2\cos^2\theta \cdot \boxed{r}\, dr\,d\theta$

D = disque rayon 2 centré en $(0,0)$

$D = \{ (x-0)^2 + (y-0)^2 \le 2^2 \} = \{ (r\cos\theta - 0)^2 + (r\sin\theta - 0)^2 \le 2^2 \}$

$= \{ x^2 + y^2 \le 4 \} = \{ r^2 \le 2^2 \} = \boxed{\{ 0 \le r \le 2 \}}$ implicitement $\theta \in [0, 2\pi]$

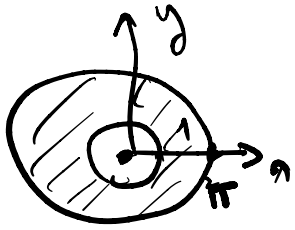
fct en r fct en θ

$= \iint_{\{0 \le r \le 2,\ 0 \le \theta \le 2\pi\}} r^3\cos^2\theta \, dr\,d\theta$ ← rectangle

$= \int_0^2 r^3\,dr \cdot \int_0^{2\pi} \cos^2\theta \, d\theta$ (période $= \frac{2\pi}{2} = \pi$)

$= \frac{1}{4}[r^4]_0^2 \cdot \left(\frac{1}{2} \times 2\pi\right) = 4 \cdot \pi = 4\pi$

(2) $D = \{ 1 \le x^2 + y^2 \le \pi^2 \}$



$x^2 + y^2 = r^2$

$$D = \{ r \ge 0,\ \theta \in [0, 2\pi] \mid 1 \le r^2 \le \pi^2 \}$$

$$= \{ r \ge 0,\ \theta \in [0, 2\pi] \mid 1 \le r \le \pi \}$$

$$= \{ 1 \le r \le \pi,\ 0 \le \theta \le 2\pi \}$$

$$\iint_D \sin(x^2 + y^2)\, dx\, dy$$

$$= \iint_{\{1 \le r \le \pi,\ 0 \le \theta \le 2\pi\}} \sin(r^2) \cdot r\, dr\, d\theta = \iint r \cdot \sin(r^2)\, dr\, d\theta$$

$\{ \dots \}$

$$= \int_1^{\pi} r \sin(r^2)\, dr \cdot \int_0^{2\pi} d\theta$$

$\int_0^{2\pi} d\theta$

$\sin(\underbrace{r^2}_{z}) \cdot \underbrace{r\, dr}_{\frac{1}{2} dz} = \frac{1}{2} dz$) changement de variable

$dz = 2r\, dr$

$z = r^2$

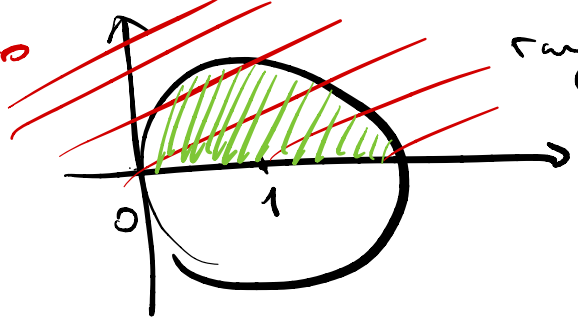
$z \in [1, \pi^2] \hookleftarrow r \in [1, \pi]$ bijectivement

$$= \int_1^{\pi^2} \sin(z) \frac{1}{2} dz \cdot 2\pi$$

$$= \frac{1}{2} \left[-\cos(z) \right]_1^{\pi^2} \cdot 2\pi$$

$$= \pi \cdot \left(\cos(1) - \cos(\pi^2) \right)$$

(3) $D =$



deux choix :

① & ② $\rightsquigarrow dxdy = rdrd\theta$

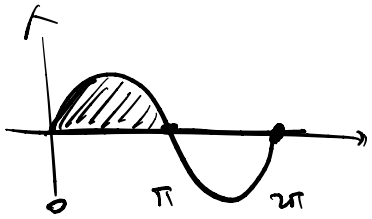
① $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\hookrightarrow \{ 0 \le \theta \le \frac{\pi}{2} \}$ $\{ 0 \le r \le 2\cos\theta \}$

② $\begin{cases} x = 1 + r\cos\theta = 1 + r\cos\theta \\ y = 0 + r\sin\theta = r\sin\theta \end{cases}$

② D = l'intersection de $\{ 0 \le r \le 1 , 0 \le \theta < 2\pi \}$

avec $\{ r\sin\theta \ge 0 \}$

$r\sin\theta \ge 0 \iff \begin{pmatrix} r > 0 , \sin\theta \ge 0 \\ \text{ou} \quad r = 0 , 0 \le \theta < 2\pi \end{pmatrix}$



$\iff \begin{pmatrix} r > 0 , 0 \le \theta \le \pi \quad (\text{ou } \theta = 2\pi) \\ \text{ou} \quad r = 0 , \quad 0 \le \theta < 2\pi \end{pmatrix}$

$\Rightarrow D = \{ 0 \le r \le 1 , \ 0 \le \theta < 2\pi \}$

$\cap \left(\{ r > 0 , 0 \le \theta \le \pi \} \cup \{ r = 0 , 0 \le \theta < 2\pi \} \right)$

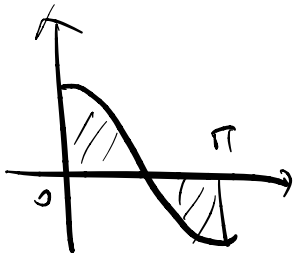
on peut ignorer lors de l'intégration

$= \{ 0 < r \le 1 , 0 \le \theta \le \pi \} \cup \{ r = 0 , 0 \le \theta < 2\pi \}$

$$\Rightarrow \iint_D x \, dx\,dy = \iint (1 + r\cos\theta) \cdot r \, dr\, d\theta$$

$$\{0 < r \le 1,\ 0 \le \theta \le \pi\}$$

$$= \int_0^1 \left(\int_0^\pi (r + r^2\cos\theta)\, d\theta \right) dr$$



$$\int_0^\pi \cos\theta \, d\theta = 0$$

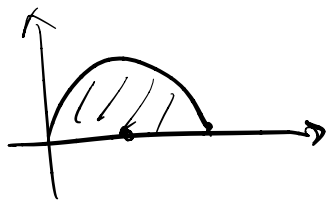
$$= \int_0^1 \left(\int_0^\pi r \, d\theta \right) dr$$

$$= \int_0^1 r \cdot \pi \cdot dr$$

$$= \left(\frac{1}{2} r^2 \right) \Big|_0^1 \cdot \pi$$

$$= \frac{1}{2} \cdot \pi$$

sens géométrique : la moyenne de x sur D · multipliée par l'aire de D.



la moyenne = 1

l'aire = $\frac{1}{2} \times \pi \cdot 1^2 = \frac{1}{2}\pi$

On a composé deux étapes de chgt de vart.

$$(x,y) \longmapsto (x',y') = (x-1, y) = (x,y) - (1,0)$$

$dx\,dy \qquad dx'\,dy'$

$$(x',y') = (r\cos\theta, r\sin\theta) \quad \text{avec} \begin{cases} 0 \le r \le 1 \\ 0 \le r\sin\theta \end{cases}$$

$$dx'\,dy' = r\,dr\,d\theta$$

(4) − (7)

(4) $\quad y \le x \iff r\sin\theta \le r\cos\theta$

$\Rightarrow \cos\theta > 0$

$(r > 0) \qquad \tan\theta \le 1$

