TD analyse 04/11/2026

calendrier :

| | |
|---|---|
| ~~04/11~~ | ~~05/11~~ |
| ~~11/11~~ | 12/11 |
| 18/11 | 19/11 |
| 23/11 | 25/11 |
| | 26/11 |
| 02/12 | 03/12 |
| 09/12 | 10/12 |

- commencer à 13h30 ?
- date du CC3 ?

Ex 4. 5. 6. 7 (1) (2)

Ex 4 changement de variable.

$$\int f(x)\,dx = \int f(\varphi(u))\,\varphi'(u)\,du$$

$f(x)$ : à rendre simple ; $\varphi(u)$ : simple

↑ rendre $f(x)$ simple avec $x = \varphi(u)$

$\log x$ $\xrightarrow{x = e^u}$ $u$

$\sqrt{1-x^2}$ $\xrightarrow{x = \sin u}$ $\pm \cos u$

(1) $\displaystyle\int \frac{\log x}{x}\,dx \overset{x = e^u}{=\!=} \int \frac{u}{e^u} \cdot e^u\,du$

$dx = (e^u)'\,du = e^u\,du$

$$= \int u \cdot du = \frac{1}{2}u^2 + C$$

$x = e^u \iff \log x = u$

$$= \frac{1}{2}(\log x)^2 + C$$

(2) $\int \sin^2 x\,dx$ ← puissance ② paire

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2}$$

$\sin x = u$ ×

$x = \arcsin u$

$$dx = \frac{1}{\sqrt{1-u^2}}\,du$$

$$\sin^2 x\,dx = \frac{u^2}{\sqrt{1-u^2}}\,du$$

$$\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx$$

$$= \int \frac{1}{2}\,dx - \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{1}{2}x - \frac{1}{2}\int \cos \underbrace{2x}_{u}\,dx$$

$2x = u$

$x = \frac{1}{2}u$

$dx = \frac{1}{2}du$

$$= \frac{1}{2}x - \frac{1}{2}\int \cos u \cdot \frac{1}{2}\,du$$

$$= \frac{1}{2}x - \frac{1}{4}\int \cos u\,du$$

$$= \frac{1}{2}x - \frac{1}{4}\sin u + C$$

$$= \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$$

(3) $\int \sin^2 x \cdot \cos x \, dx$ $\quad (\sin/\cos)^{\text{impaire}}$

(a) $u = \sin x \Rightarrow \cos x = \sqrt{1-\sin^2 x}$
$= \sqrt{1-u^2}$.

$x = \arcsin u$

$dx = \frac{1}{\sqrt{1-u^2}} du$

$\Rightarrow \int \sin^2 x \cdot \cos x \, dx = \int u^2 \cdot \sqrt{1-u^2} \cdot \frac{1}{\sqrt{1-u^2}} du$

$= \int u^2 du$

$= \frac{1}{3} u^3 + C$

$= \frac{1}{3} \sin^3 x + C$

(b) $\int \sin^2 x \cos x \, dx = \int u(x)^2 u'(x) dx$

$u = \sin x$ $\quad \Rightarrow \int u^2 du$

$du = \cos x \, dx$

$u'(x) = \cos x$

(4) $\int \sqrt{1-x^2}\,dx$

$1 = \sin^2 u + \cos^2 u$

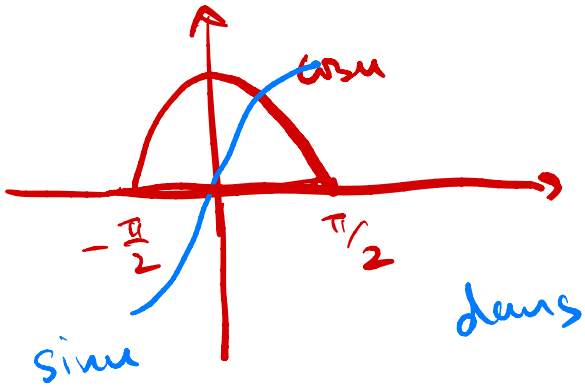
$\sqrt{1-\sin^2 u} = \pm \cos u$

$x = \sin u \Rightarrow \int (\pm \cos u)\cdot \cos u\,du$

$dx = \cos u\,du$

↑ prenons "+", cela veut dire que :

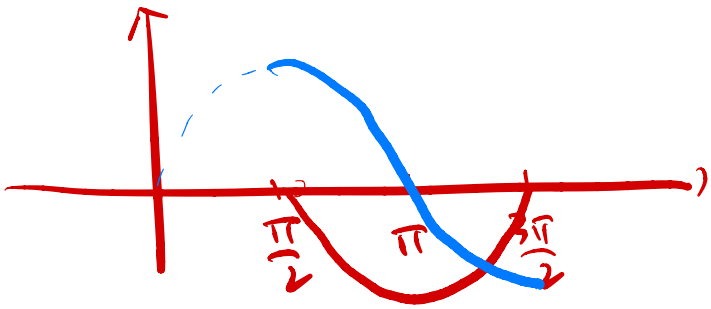
$\cos u \geq 0 \iff u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$




sin(u) parcourt toutes les valeurs dans l'intervalle $[-1,1]$

( si on prend "$-\cos u = \sqrt{1-\sin^2 u}$",

c'est $\iff \cos u \leq 0 \iff u \in [\frac{\pi}{2}\ \frac{3\pi}{2}]$




)

$\sqrt{1-x^2} = +\cos u$

$$\int \sqrt{1-x^2}\,dx = \int \cos^2 u\,du$$

$$= \int \frac{1}{2}(1+\cos 2u)\,du$$

$$\overset{(2)}{=} \frac{1}{2}u + \frac{1}{2}\times\frac{1}{2}\sin 2u + C$$

$u \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ $= \frac{1}{2}u + \frac{1}{4}\sin 2u + C$

$x = \sin u \Leftrightarrow \arcsin x = u$

$\sqrt{1-x^2} = +\cos u$ $= \frac{1}{2}\arcsin x + \frac{1}{4} \cdot 2 \cdot \sin u \cos u$

$\sin u = x$, $\cos u = +\sqrt{1-x^2}$

$= \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C$

**Rmq:** si on prend $\sqrt{1-x^2} = -\cos u$,

$$\int \sqrt{1-x^2}\,dx = -\int \cos^2 u\,du$$

$$= -\int \frac{1}{2}(1+\cos 2u)\,du$$

$x = \sin u \not\Rightarrow \arcsin x = u$ $= -\frac{1}{2}u - \frac{1}{4}\sin 2u + C$

si $u \in [\frac{\pi}{2}, \frac{3\pi}{2}]$

$\sqrt{1-x^2} = -\cos u$ $= -\left(\frac{1}{2}\pi - \frac{1}{2}\arcsin(x)\right)$

mais $u = \pi - \arcsin(x)$ $- \frac{1}{2}x(-\sqrt{1-x^2}) + C$

$= \frac{1}{2}\arcsin(x) + \frac{1}{2}x\sqrt{1-x^2} + C$

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\arcsin(x) + \frac{1}{2}x\sqrt{1-x^2} + C$$

(5) $\int \sqrt{4-x^2}\,dx$ $\quad 4-x^2 = 4-4u^2$

$= \int \sqrt{4-4u^2}\cdot 2\,du$ $\quad x^2 = 4u^2$

$\boxed{x = 2u}$

$= \int 2\sqrt{1-u^2}\cdot 2\,du$ $\quad dx = 2\,du$

$= 4\int \sqrt{1-u^2}\,du$

$\overset{(4)}{=} 2\arcsin(u) + 2u\sqrt{1-u^2} + C$

$\overset{u=\frac{x}{2}}{=} 2\arcsin\left(\frac{x}{2}\right) + x\sqrt{1-\frac{x^2}{4}} + C$

$x\sqrt{1-\frac{x^2}{4}} = \frac{1}{2}x\sqrt{4-x^2}$

(6) $\int \frac{1}{7+\tan x}\,dx$ $\quad u = \tan x$

↓

- rendre dénom. plus simple.
- $x(u) = \arctan(u)$ sa dérivée "simple"

$dx = x'(u)\,du$

$$= \int \frac{1}{7+u} \cdot \frac{1}{1+u^2}\,du$$

appliquer la méthode de Ex 3

$$= \int \left( \frac{A}{7+u} + \frac{Bu+C}{1+u^2} \right) du$$

$$= \int \frac{1}{50} \left( \frac{1}{7+u} - \frac{u-7}{u^2+1} \right) du$$

$(u^2+1)' = 2u$

$$= \int \frac{1}{50} \left( \frac{1}{7+u} - \frac{\frac{1}{2}\cdot 2u}{u^2+1} + \frac{7}{u^2+1} \right) du$$

$\ln(7+u)$ $\ln(u^2+1)$ $\arctan(u)$

$$= \frac{1}{50} \ln(7+u) - \frac{1}{100} \ln(u^2+1) + \frac{7}{50} \arctan(u)\,du + C$$

$x = \arctan(u)$

$\tan x = u$

$$= \frac{1}{50}\ln(|7+\tan x|) - \frac{1}{100}\ln(\underbrace{\tan^2 x + 1}_{\frac{\sin^2 x + \cos^2 x}{\cos^2 x}}) + \frac{7}{50}x + C$$

$$= \frac{1}{\cos^2 x}$$

(après simplification)

$$= \frac{1}{50}\ln\left(\left|\frac{7\cos x + \sin x}{\cos x}\right|\right) - \frac{1}{100}\ln\left(\frac{1}{\cos^2 x}\right) + \frac{7}{50}x + C$$

$$-\frac{1}{100}\left(-\ln\cos^2 x\right)$$

$$= -\frac{1}{100}\left(-2\ln|\cos x|\right)$$

$$= \frac{1}{50}\ln|\cos x|$$

$$= \frac{1}{50}\ln\left(\left|\frac{7\cos x + \sin x}{\cos x}\right| \cdot |\cos x|\right) + \frac{7}{50}x + C$$

$$= \frac{1}{5} \ln\left(|7\cos x + \sin x|\right) + \frac{7}{5}x + C$$

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Souvent : $\int \frac{f(x)}{1+g(x)}\,dx$, $\int \sqrt{1-g^2(x)}\,dx$

essayer : $g(x) = u$

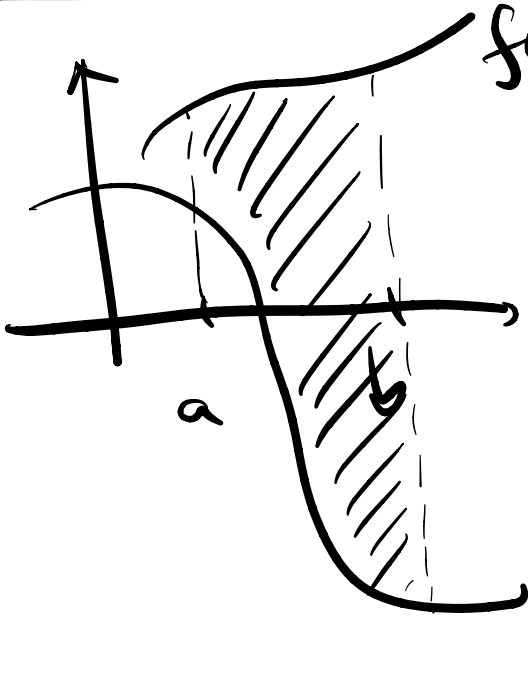
$\rightsquigarrow x = g^{-1}(u)$

$dx = (g^{-1})'(u)\,du$

.....

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Exs



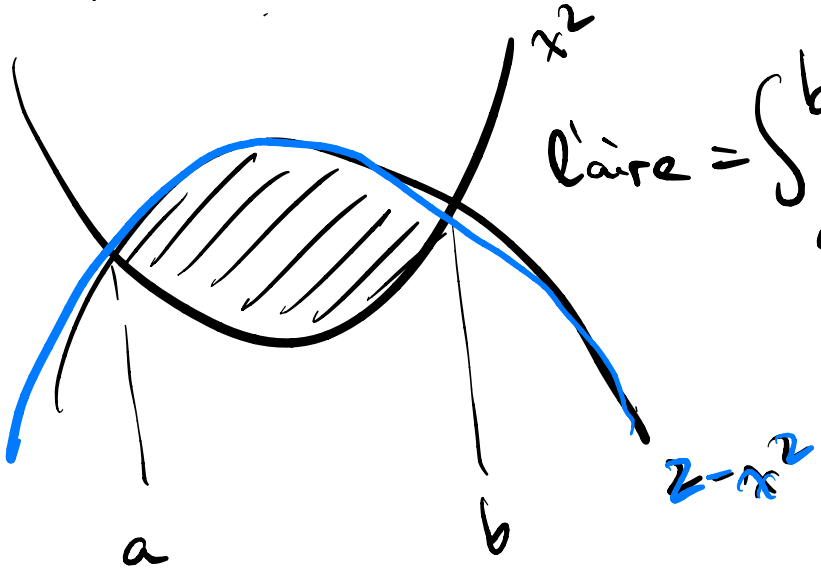

$f(x) \geq g(x)$

son aire

$= \int_a^b \left(f(x) - g(x)\right)dx$

$\geq 0$

$y = x^2$ $\quad y = 2 - x^2$

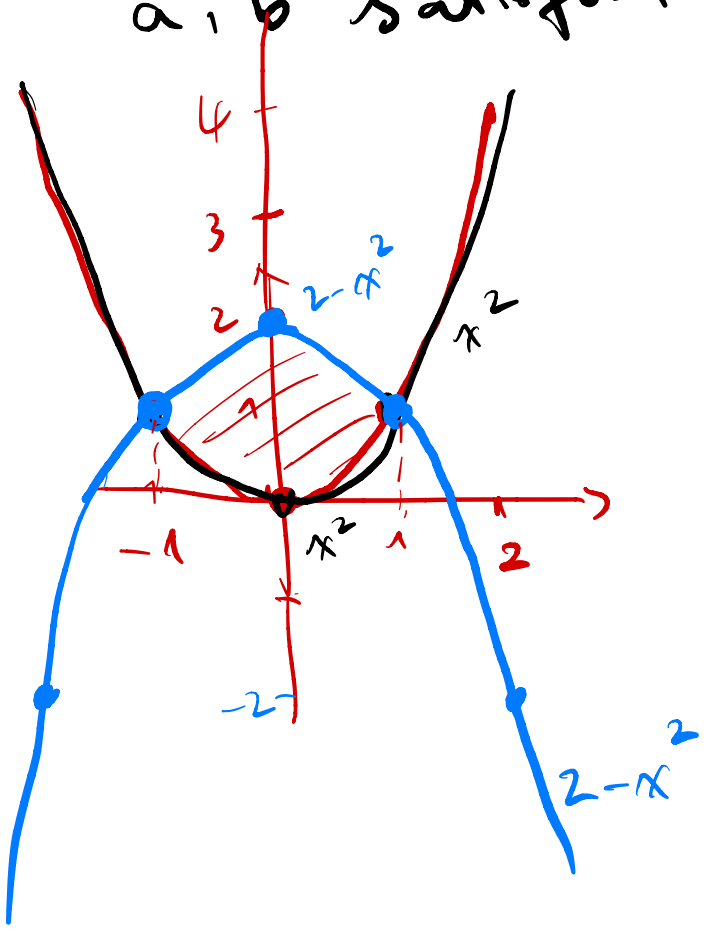



$$\text{l'aire} = \int_a^b \left( (2-x^2) - x^2 \right) dx$$

$a, b$ satisfont l'équation : $x^2 = 2 - x^2$

$$\Rightarrow 2x^2 = 2$$

$$\Rightarrow x = \pm 1.$$




$$\text{l'aire} = \int_{-1}^{1} \left( 2 - x^2 - x^2 \right) dx$$

$$= \int_{-1}^{1} \left( 2 - 2x^2 \right) dx$$

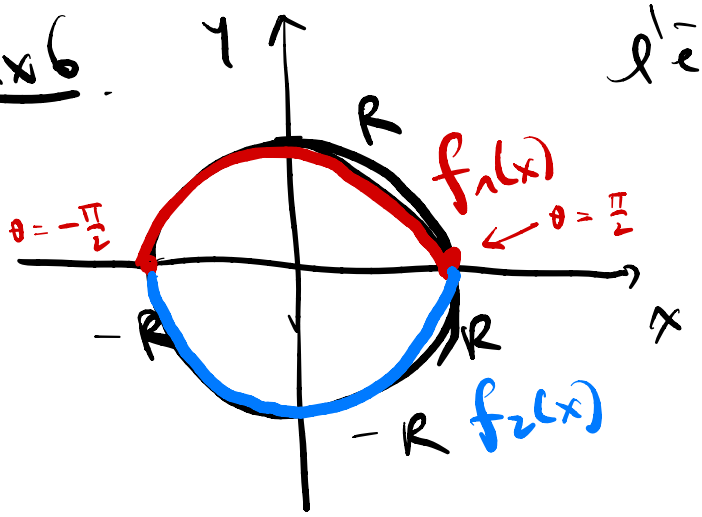
$$= \left( 2x - \frac{2}{3}x^3 \right) \Big|_{-1}^{1}$$

$$= \left( 2 - \frac{2}{3} \right) - \left( -2 + \frac{2}{3} \right)$$

$$= \frac{8}{3}$$

$x = \sin\theta$

**Ex 6.**




l'éq d'un disque centré en O :

$$x^2 + y^2 = R^2$$

exprimer $y$ en termes de $x$ :

$$y = \pm\sqrt{R^2 - x^2}$$

$+\sqrt{R^2-x^2} = f_1(x)$

$-\sqrt{R^2-x^2} = f_2(x)$

$$\text{l'aire} = \int_{-R}^{R} \left(f_1(x) - f_2(x)\right) dx$$

on peut aussi calculer :

$$\text{l'aire} = 2 \times \left(\text{l'aire du demi-disque du haut}\right)$$

$$= 2\int_{-R}^{R} f_1(x)\, dx$$

$\Rightarrow$ l'aire du disque de rayon R

$$= 2\int_{-R}^{R} \sqrt{R^2 - x^2}\, dx$$

$-R \le x \le R$

$\rightsquigarrow$ Ex 4 (4), (5)

$x = 2u$

$x = R \cdot u$ $\quad \sqrt{R^2 - x^2}\, dx$ $\quad -R \le Ru \le R$ $\Leftrightarrow$ $-1 \le u \le 1$

$$= \sqrt{R^2 - R^2u^2} \cdot R\,du$$

$$= \sqrt{R^2} \cdot \sqrt{1-u^2} \cdot R\,du$$

$$= R^2 \sqrt{1-u^2}\,du$$

$$\Rightarrow \text{l'aire} = 2\int_{-1}^{1} R^2\sqrt{1-u^2}\,du$$

$$= 2R^2 \int_{-1}^{1} \sqrt{1-u^2}\,du$$

$$= 2R^2 \left[ \frac{1}{2}\arcsin(u) + \frac{1}{2}u\sqrt{1-u^2} \right]\Bigg|_{-1}^{1}$$

$$= 2R^2\left( \frac{1}{2}\frac{\pi}{2} - \frac{1}{2}\left(-\frac{\pi}{2}\right)\right) = 2R^2 \cdot \frac{\pi}{2}$$

$$\boxed{\int \sqrt{1-u^2}\,du = \frac{1}{2}\arcsin(u) + \frac{1}{2}u\sqrt{1-u^2} + C}$$

$u$

$\pi R^2$

Rmq : on peut calculer en un seul temps :

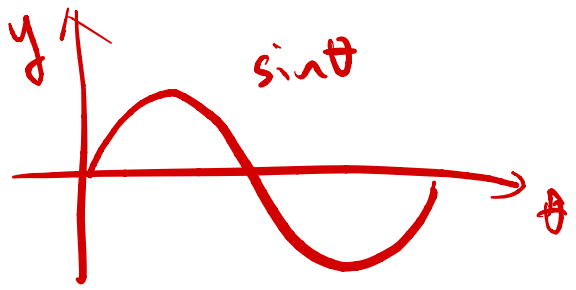
$$2\int_{-R}^{R} \sqrt{R^2 - x^2}\,dx \qquad -R \le x = R\sin\theta \le R$$

$$-1 \le \sin\theta \le 1$$

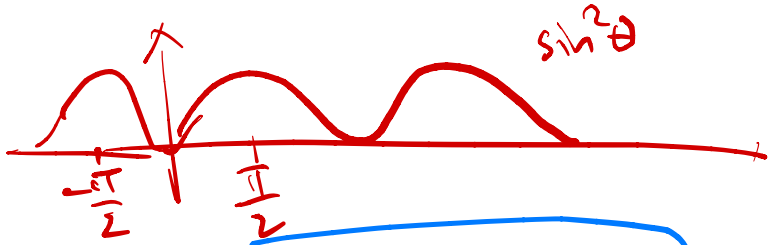
$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R\cos\theta \cdot R\cos\theta\,d\theta$$

$$= 2R^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta\,d\theta = 2R^2 \cdot \frac{\pi}{2} = \pi R^2$$




$2\pi$ - périodique




$\pi$ - périodique

$$\Rightarrow \boxed{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta \, d\theta} = \int_0^{\pi} \cos^2\theta \, d\theta$$

$\frac{\pi}{2}$ = Longueur $\pi$

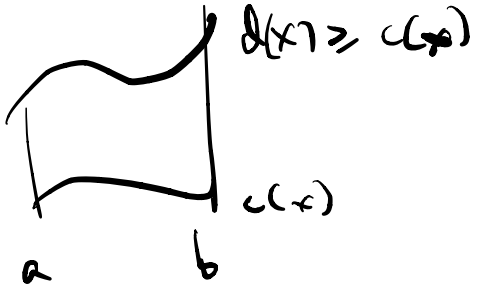
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\left(\theta + \tfrac{\pi}{2}\right) d\theta$$

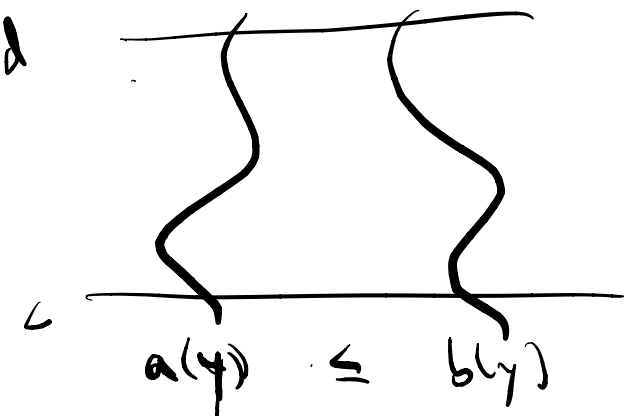
$$= \boxed{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \, d\theta}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos^2\theta + \sin^2\theta\right) d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 \cdot d\theta = \pi$$

**Ex 7**

$$\iint_D f(x,y)\, dx\, dy$$



$$\{ a \le x \le b,\ c(x) \le y \le d(x) \}$$

ou inversement sur les rôles de $x$ & $y$.