A NOTE ON THE CONDITION $(\dagger)_A$ OF KERZ–SAITO–TAMME

YICHENG ZHOU

Abstract. In this short note, we study the condition $(\dagger)_A$ that appeared in the work of Kerz–Saito–Tamme [KST19]. We propose a variant in the non-Noetherian setting, which concerns specifically rigid-analytic varieties over a (not necessarily discretely valued) nonarchimedean field k. We explain that most arguments of Kerz–Saito–Tamme go through verbatim, conditionally on a pro-vanishing result.

Contents

1. The $\mathrm{N}^\pi\mathrm{K}$ vanishes in the semistable case 2
1.1. Definitions and main theorem 2
1.2. Regular case 3
1.3. Examples 4
1.4. Standard semistable case 7
1.5. Semistable case 9
1.6. Semistable formal case 12
2. Dagger condition in Kerz–Saito–Tamme's work 14
2.1. Algebraic and continuous K-theory of Tate rings 14
2.2. Analytic K-theory of adic rings and Tate rings 15
2.3. Analytic KV-theory of Tate rings 18
2.4. Application: representability of analytic K-theory 19
References 20

This note is originated in an earlier discussion with Christian Dahlhausen and Can Yaylali in August 2026 regarding an attempt to remove the Noetherian assumption in Kerz–Saito–Tamme's works [KST19, KST23]. For our purpose, we restrict ourselves to the context of rigid-analytic varieties over a nonarchimedean field k (or related formal or algebraic schemes over k°) which is not necessarily discretely valued. In this context, the Noetherian assumption in [KST23] concerning the analytic descent of the analytic K-theory can be removed by comparing to the nuclear K-theory, as treated in [DYZ26]; there is even Nisnevich descent. In this note, we will focus on the dagger condition $(\dagger)_A$ in [KST19]; we will provide an extension of the dagger condition (see 1.6.4). The structure of this note is as below: we will prove a pro-homotopy invariance in Section 1 under suitable mild singularities condition, and then point out in Section 2 that Kerz–Saito–Tamme's works *op. cit.* remain valid under this generalised dagger condition. Emphasis will be put on explaining where this technical condition was involved *op. cit.*

Note that if k is discretely valued, then we are in the Noetherian case, which has been dealt with in *op. cit.*

Acknowledgements. The author would like to thank Christian Dahlhausen and Heidelberg University for their invitation and hospitality, which have facilitated discussions with CD and Can Yaylali from which

Date: 19th October 2026.

the author has very much benefited. The author is funded by the Fondation Mathématique Jacques Hadamard (FMJH).

1. The $\mathrm{N}^\pi\mathrm{K}$ vanishes in the semistable case

Let k be a nonarchimedean field, i.e. a complete non-trivially real-valued field, whose ring of integer is denoted by k°, with maximal ideal $k^{\circ\circ} = \mathfrak{m}$. Here, the non-triviality of valuation implies the existence of a topologically nilpotent unit $\pi \in k^\circ$, which we call a *pseudo-uniformiser*[1], which we shall not fix but which will be clear from the context.

Part of the statements remain true for any pair $(\mathcal{O}, \pi)$ where $\mathcal{O}$ is a valuation ring and $\pi \in \mathcal{O}$ is non-zero. The statements will be trivial is π is a unit in $\mathcal{O}$, hence we will assume π to be a non-unit. When $\mathcal{O}$ is not of rank one, the choice of π matters, but only to the extent of a choice of its associated radical ideal $\sqrt{\pi\mathcal{O}}$.

There are two (pro-)spectra-valued constructions on rings, $\mathrm{NK}(-)$ and $\mathrm{N}^\pi\mathrm{K}(-)$, which measure the defects of $\mathbf{A}^1$-homotopy invariance and, respectively, of certain "pro"-$\mathbf{A}^1$-homotopy invariance (which is related to the analytic $\mathbf{A}^1$-homotopy invariance), of rings (see Definition 1.1.2).

Our main goal of this section is to show that $\mathrm{N}^\pi\mathrm{K}(X) \simeq 0$ for strictly semistable schemes X over k° (Corollary 1.5.6) and to show a (partially) completed version of it (Proposition 1.6.7). When the valuation on k is discrete and π is a *uniformiser*, then X is regular and we have the stronger vanishing $\mathrm{NK}(X) \simeq 0$ by a classical result of Weibel [Wei13, Theorem V.6.3].

Moreover, as the proof strategy will demonstrate, the condition of being "semistable" can be relaxed to certain extent, for example, to being "polystable" (see Remark 1.5.7 (i)) or being "with toric l.c.i. singularities" (see Remark 1.5.7 (ii)). These generalisations are a matter of conclusion from rather than essential improvement of the results ; I retain my efforts from adapting them to specific applications.

1.1. Definitions and main theorem.

1.1.1. **Remark.** Here, contrary to usual convention of semistability, we adopt a weaker convention, following [Tem17], that π is not necessarily a uniformiser of k, which is very practical in the general setting where k is not discretely valued.

1.1.2. **Definition.** (i) For any ring R, define the spectra

$$\mathrm{NK}(R) \coloneqq \mathrm{fib}(\mathrm{K}(R[T]) \to \mathrm{K}(R)).$$

(ii) Let $\mathcal{O}$ be a valuation ring and $\pi \in \mathcal{O}$ which is non-zero and non-unit. For any $\mathcal{O}$-algebra R, define the pro-spectra

$$\mathrm{N}^\pi\mathrm{K}(R) \coloneqq \mathrm{fib}(\text{“}\lim_{T\mapsto\pi T}\text{”}\,\mathrm{K}(R[T]) \to \mathrm{K}(R)).$$

This definition extends to arbitrary functor F on the category of (commutative rings) valued in (pro-)spectra.

1.1.3. **Remark.** If $\mathcal{O} = k^\circ$ for a nonarchimedean field k,which has rank one valuation, the definition $\mathrm{N}^\pi\mathrm{K}(R)$ is independent of the choice of the pseudo-uniformiser π.

In general, $\mathrm{N}^\pi\mathrm{K}(R)$ is well-defined up to equivalence in $\mathrm{Pro(Sp)}$ depends only on the ideal $\sqrt{\pi\mathcal{O}}$ of $\mathcal{O}$.

1.1.4. **Lemma.** *We have*

$$\mathrm{K}(R[T]) \simeq \mathrm{K}(R) \oplus \mathrm{NK}(R) \quad \textit{in } \mathrm{Sp},$$
$$\text{“}\lim_{T\mapsto\pi T}\text{”}\,\mathrm{K}(R[T]) \simeq \mathrm{K}(R) \oplus \mathrm{N}^\pi\mathrm{K}(R) \quad \textit{in } \mathrm{Pro(Sp)}.$$

[1]The terminology of *pseudo-* reflects the fact that it is not necessarily a uniformiser, i.e. the maximal ideal $\mathfrak{m}$ of k° is not the principal ideal πk°; however, we have $\mathfrak{m} = \sqrt{\pi k^\circ}$.

Proof. Both arrows are induced by the map $T \mapsto 0$, which has a canonical splitting induced by the natural embedding $R \mapsto R[T]$. □

1.1.5. **Lemma.** *We have* $\mathrm{N}^\pi\mathrm{K}(R) \simeq$ "lim"$_{T\mapsto\pi T}\,\mathrm{NK}(R)$ *in* Pro(Sp).

Proof. It is clear from the definition. □

1.1.6. **Lemma.** *If R has bounded π^∞-torsion, then* $\mathrm{N}^\pi\mathrm{K}(R) \simeq 0$ *in* Pro(Sp).

Proof. If R has bounded π^∞-torsion, then the filtered system $(R[T])_{T\mapsto\pi T}$ is equivalent to the constant system R. In particular, $\mathrm{N}^\pi\mathrm{K}(R) \simeq 0$ as pro-spectra. □

1.1.7. **Remark.** The analogue vanishing for $\mathrm{NK}(R)$ fails in general. For example, we have $\mathrm{NK}(R) \not\simeq 0$ if $R = \mathcal{O}/\pi$ and $\pi\mathcal{O}$ is not a prime ideal[2]. Indeed, one can then choose a minimal prime ideal $\mathfrak{p}$ over $\pi\mathcal{O}$ and an element $a \in \mathfrak{p}\backslash\pi\mathcal{O}$, and then $1 + aT \in R[T]^\times$ represents a non-zero class in $\mathrm{NK}(R)$.

1.1.8. **Definition.** Let $\mathcal{O}$ be a valuation ring and $\pi \in \mathcal{O}$ be non-zero and non-unit. An $\mathcal{O}$-scheme X is called *strictly π-semistable* if, Zariski-locally, it is étale over

$$A_{d,r} = k^\circ[t_0, t_1, \dots, t_d]/(t_0 \cdots t_r - \pi'), \tag{1.1.9}$$

where π' is a non-zero element of $\sqrt{\pi\mathcal{O}}$ and $0 \le r \le d$.

An $\mathcal{O}$-algebra A is called *strictly π-semistable* if $\mathrm{Spec}(A)$ is a strictly π-semistable $\mathcal{O}$-scheme.

We make similary definitions for formal schemes, replacing $\mathcal{O}[t_0, t_1, \dots, t_d]$ with the ring of π-adically convergent power series.

In the case where $\mathcal{O} = k^\circ$, we recover the usual definitions of (strict) semistability.

In order to formulate a variant of [KST19, Lemma 5.13 (ii)], we wish to prove the following theorem.

Theorem (Corollary 1.5.6)**.** *For any strictly π-semistable k°-algebra A, the pro-spectra* $\mathrm{N}^\pi\mathrm{K}(A)$ *is weakly contractible, i.e.* $\mathrm{N}^\pi\mathrm{K}(A)$ *becomes equivalent to zero in* $\mathrm{Pro}(\mathrm{Sp}^+)$.

Then it will be reasonable to consider a variant of the dagger condition (see 1.6.4) and prove a variant of [KST19, Lemma 5.13 (ii)] (see Proposition 1.6.7), which is the key to the work of Kerz–Saito–Tamme [KST19].

1.2. **Regular case.** Here we provide some recollections on the $\mathbf{A}^1$-homotopy invariance in the regular case, where we consider the notion of regularity for non-Noetherian rings (see Definition 1.2.3) because we want to go beyond Noetherian cases.

Let us start with the baby case of the $\mathcal{O}$-algebra $\mathcal{O}$. Recall the following theorem:

1.2.1. **Theorem** (Kelly–Morrow, [KM21, Theorem 3.3])**.** *Let $\mathcal{O}$ be a valuation ring. Then*

(i) $\mathrm{K}(\mathcal{O}) \to \mathrm{K}(\mathcal{O}[T_1, \dots, T_d])$ *is an equivalence for any $d \ge 1$;*
(ii) $\mathrm{K}(\mathcal{O}) \to \mathrm{KH}(\mathcal{O})$ *is an equivalence;*
(iii) $\mathrm{K}_n(\mathcal{O}) = 0$ *for all $n < 0$.*

In particular, $\mathrm{NK}(\mathcal{O}) \simeq 0$.

Let us recall its slight generalisation:

1.2.2. **Theorem** (Antieau–Mathew–Morrow, [AMM22, Proposition 2.4])**.** *Let $\mathcal{O}$ be a valuation ring. Let A be a smooth $\mathcal{O}$-algebra. Then*

(i) $\mathrm{K}(A) \to \mathrm{K}(A[T_1, \dots, T_d])$ *is an equivalence for any $d \ge 1$;*
(ii) $\mathrm{K}(A) \to \mathrm{KH}(A)$ *is an equivalence;*
(iii) $\mathrm{K}_n(A) = 0$ *for all $n < 0$.*

[2]In a valuation ring $\mathcal{O}$, the radical ideal $\sqrt{\pi\mathcal{O}}$ is always a prime ideal. Hence, the condition of $\pi\mathcal{O}$ being prime amounts to asking that $\pi\mathcal{O}$ is radical.

In particular, $\mathrm{NK}(A) \simeq 0$.

The proof of it has actually been given Antieau–Mathew–Morrow [AMM22]. Let us recall it for our own convenience, as this short note is not intended to be too concise.

1.2.3. **Definition** (Regularity of non-Noetherian rings)**.** Let R be a ring.

(i) R is called *coherent* if every finitely generated ideal of it is a finitely presented module.
(ii) R is *regular* if every finitely generated ideal (or equivalently, module) of it has finite projective dimension [Gla89, §6.2 Definition, Theorem 6.2.1].
(iii) R is called *stably coherent* if every polynomial extension of it (i.e. $R[T_1, \ldots, T_d]$ with $d \geq 1$) is coherent.

1.2.4. **Lemma** ([Gla89, Theorem 6.2.5], [GL24, Lemma A.1])**.** *Let $R \to R'$ be a faithfully flat ring map. If R' is coherent (*resp. *stably coherent, coherent regular, stably coherent regular), so is R.* □

1.2.5. **Lemma** ([GL24, Lemma A.2])**.** *Let $R \to R'$ be an étale ring map. If R is coherent (*resp. *stably coherent, coherent regular, stably coherent regular), so is R'. The converse is true if the map is an étale covering (or equivalently, faithfully flat).* □

1.2.6. **Lemma** ([Gla89, Corollary 6.2.14])**.** *If R is stably coherent and regular, then every polynomial extension of R is regular.* □

1.2.7. **Remark.** Coherence is not necessarily stable under polynomial extensions.

1.2.8. **Corollary.** *Let $R \to R'$ be a smooth ring map with R stably coherent. If R is regular, so is R'. The converse is true if the map is a smooth covering (or equivalently, faithfully flat).*

Proof. Any smooth ring map is étale locally on the source and on the target an étale map from a polynomial algebra. The forward implication follows then from Lemma 1.2.5 and Lemma 1.2.6.

The converse follows directly from Lemma 1.2.4. □

Now we can state the key result for proving Theorem 1.2.2.

1.2.9. **Proposition.** *Every valuation $\mathcal{O}$ is stably coherent regular.*

Proof. It is known that $\mathcal{O}$ is stably coherent [Gla89, Theorem 7.3.3].

Let us show that $\mathcal{O}$ is regular. Valuation rings have weak global dimension at most one since any finitely generated ideal of $\mathcal{O}$ is principal [Gla89, p. 25], and this dimension bounds the projective dimension of all finitely presented ideals (or equivalently, modules) [Gla89, Corollary 2.5.5, Corollary 2.5.6]. By definition, we see that $\mathcal{O}$ is regular. □

Proof of Theorem 1.2.2. [3] As explained by Kelly–Morrow in [KM21, Theorem 3.3], the assertions (i)–(iii) are consequences of the stably coherent regularity of A, which we claim now to be true[4]. By the smooth-local nature of coherent regularity (Corollary 1.2.8), one reduces to the case $A = \mathcal{O}$. Then it is Proposition 1.2.9. □

1.3. **Examples.** Let $\mathcal{O}$ be a valuation ring. In some particular statements below, it might to be necessary to impose $\mathcal{O} = k^\circ$ specifically.

Our goal is to generalise Theorem 1.2.2 to the semistable case.

[3] While the proof of Kelly–Morrow [KM21, Theorem 3.3] reduces to the case with certain countability assumption, we follow instead more closely that of Banerjee–Sadhu [BS22, Theorem 1.1], with slight modifications. It turns out that the proof coincides with that of [AMM22, Proposition 2.4].

[4] This is actually the statement of [AMM22, Corollary 2.3]. But we give a proof of it below. At the moment when the proof below was written, we were unaware of *loc. cit.*, knowing only that it could be very elementary and classical but lacks literature.

(A). *Nodal algebras.* Consider first the case of the standard nodal algebra

$$A^{\mathrm{nodal}} \coloneqq \mathcal{O}[t_0, t_1]/(t_0 t_1 - \pi),$$

where $\pi \in k^\circ$ is a pseudo-uniformiser (1.1.9). Étale algebras over A^{nodal}, which provide étale local models of semistable models over k°, will be studied next but one.

1.3.1. **Remark.** If $\mathcal{O}$ is a discrete valuation ring, then A^{nodal} is regular if and only if π is a uniformiser. If $\mathcal{O}$ is the ring of integers k° where k is non-discretely valued, then A^{nodal} is never regular (in the sense of Definition 1.2.3). In fact, this is because A^{nodal} is non-Noetherian and $A^{\mathrm{nodal}} \otimes_{k^\circ} \tilde{k}$ is singular, and by applying [GL24, Proposition 3.9 (ii)][5].

It is no surprise then to see that Theorem 1.2.2 could fail for nodal algebras.

1.3.2. **Theorem.** *Let $\mathcal{O}$ be a valuation ring and $\pi \in \mathcal{O}$ which is non-zero and not a unit. For any smooth $\mathcal{O}$-algebra B, there is a canonical equivalence of spectra*

$$\mathrm{NK}(A^{\mathrm{nodal}} \otimes_{\mathcal{O}} B) \simeq \mathrm{NK}(B/\pi) \tag{1.3.3}$$

and isomorphisms

$$\mathrm{K}_n(A^{\mathrm{nodal}} \otimes_{\mathcal{O}} B) \simeq \mathrm{K}_n(B/\pi), \quad n < 0. \tag{1.3.4}$$

As a result, for A^{nodal}, the $\mathbf{A}^1$-homotopy invariance does not hold in general, while the negative K-groups do vanish. Instead, the *"pro"-$\mathbf{A}^1$-homotopy invariance*, i.e. the contractibility of $\mathrm{N}^\pi\mathrm{K}$, holds.

1.3.5. **Corollary.** *Let $\mathcal{O}$ be a valuation ring and $\pi \in \mathcal{O}$ which is non-zero and not a unit.*

(i) $\mathrm{NK}(A^{\mathrm{nodal}}) \simeq 0$ *if and only if $\pi\mathcal{O} \subset \mathcal{O}$ is a prime ideal;*
(ii) $\mathrm{N}^\pi\mathrm{K}(A^{\mathrm{nodal}}) \simeq 0$ *in* $\mathrm{Pro(Sp)}$*;*
(iii) $\mathrm{K}_0(A^{\mathrm{nodal}}) \simeq \mathbf{Z}$*, and* $\mathrm{K}_n(A^{\mathrm{nodal}}) = 0$ *for all $n < 0$;*
(iv) $\mathrm{NK}_n(A^{\mathrm{nodal}}) = 0$ *for all $n \leq 0$.*

More generally, for any smooth $\mathcal{O}$-algebra B,

(i') $\mathrm{NK}(A^{\mathrm{nodal}} \otimes_{\mathcal{O}} B) \simeq 0$ *if and only if $\pi\mathcal{O} \subset \mathcal{O}$ is a prime ideal;*
(ii') $\mathrm{N}^\pi\mathrm{K}(A^{\mathrm{nodal}} \otimes_{\mathcal{O}} B) \simeq 0$ *in* $\mathrm{Pro(Sp)}$*;*
(iii') $\mathrm{K}_n(A^{\mathrm{nodal}} \otimes_{\mathcal{O}} B) = 0$ *for all $n < 0$;*
(iv') $\mathrm{NK}_n(A^{\mathrm{nodal}} \otimes_{\mathcal{O}} B) = 0$ *for all $n \leq 0$.*

Proof. According to Theorem 1.3.2, we have $\mathrm{NK}(A^{\mathrm{nodal}}) \simeq \mathrm{NK}(\mathcal{O}/\pi)$ and $\mathrm{K}_n(A^{\mathrm{nodal}}) \simeq \mathrm{K}_n(\mathcal{O}/\pi)$ for $n < 0$.

(i) The "only if" direction follows from Remark 1.1.7. For the "if" direction, if $\pi\mathcal{O}$ is a prime ideal, Then $\mathcal{O}/\pi$ is still a valuation ring, so that $\mathrm{NK}(\mathcal{O}/\pi)$ is contractible by Theorem 1.2.1.

(ii) This follows from Lemma 1.1.6.

(iii) This is a consequence of the nil-invariance of non-positive K-groups, noticing that $\sqrt{\pi\mathcal{O}}$ is always a prime ideal. More precisely, $(\mathcal{O}/\pi)_{\mathrm{red}}$ is the filtered colimit of nil-thichekings $\mathcal{O}/I$ of $\mathcal{O}/\pi$. By nil-invariance of negative K-groups, we obtain $\mathrm{K}_n((\mathcal{O}/\pi)_{\mathrm{red}}) = \mathrm{colim}_I \mathrm{K}_n(\mathcal{O}/I) \simeq \mathrm{K}_n(\mathcal{O}/\pi)$ for $n \leq 0$. Also, $(\mathcal{O}/\pi)_{\mathrm{red}} \simeq \mathcal{O}/\sqrt{\pi\mathcal{O}}$ is a valuation ring, so $\mathrm{K}_0((\mathcal{O}/\pi)_{\mathrm{red}}) \simeq \mathbf{Z}$ and $\mathrm{K}_n((\mathcal{O}/\pi)_{\mathrm{red}}) = 0$ for $n < 0$.

(iv) By nil-invariance of non-positive K-groups, we have $\mathrm{NK}_n(A^{\mathrm{nodal}}) \simeq \mathrm{NK}_n((\mathcal{O}/\pi)_{\mathrm{red}})$ for $n \leq 0$, as in (iii), which vanishes by Theorem 1.2.1.

For (i')–(iv'), the proof is the same. □

Let us come back to the proof of Theorem 1.3.2. As A^{nodal} is not regular (Remark 1.3.1) in general (if $\pi \in \mathcal{O}$ is not a uniformiser), we are not able to proceed similarly as for Theorem 1.2.2. Instead, we will locate $\mathrm{Spec}(A^{\mathrm{nodal}})$ as an affine open in a quasi-smooth blow-up, and resort to the blow-up formula for algebraic K-theory.

[5]Maybe it is unnecessary to refer to this as there exists perhaps an simpler argument.

1.3.6 (Quasi-smooth blow-ups and blow-up formula for algebraic K-theory). Recall that a closed immersion $Z \subset X$ is called *quasi-smooth* if Zariski-locally on X, Z is defined by a regular sequence $f_1, \dots, f_c \in \Gamma(X, \mathcal{O}_X)$; we then say that this is a closed immersion of codimension c. In this case, we have a blow-up formula proved by Thomason [Tho93, Théorème 2.1] (or see [Kha20, Corollary 4.4]):

$$\mathrm{K}(\mathrm{Bl}_Z X) \simeq \mathrm{K}(X) \oplus \bigoplus_{k=1}^{c-1} \mathrm{K}(Z). \tag{1.3.7}$$

This holds after any flat base change, since blow-up commutes with flat base change [Sta18, Lemma 0805].

Proof of Theorem 1.3.2. Consider the scheme $X := \mathrm{Spec}(\mathcal{O}[u])$ and the locus $Z := V((u,\pi)) = \mathrm{Spec}(\mathcal{O}/\pi)$. The closed immersion $Z \hookrightarrow X$ is quasi-smooth of codimension two as it is defined by the regular sequence u, π of $\mathcal{O}[u]$. Let $\widetilde{X}$ be the blow-up $\mathrm{Bl}_Z X$. It is covered by two affine opens

$$\begin{aligned} D_+(u) &\simeq \mathrm{Spec}\Big(\mathcal{O}[u][\frac{\pi}{u}]\Big) \simeq \mathrm{Spec}(\mathcal{O}[u,\bar{u}]/(u\bar{u}-\pi)) \simeq \mathrm{Spec}(A^{\mathrm{nodal}}), && \bar{u} = \frac{\pi}{u}, \\ D_+(\pi) &\simeq \mathrm{Spec}\Big(\mathcal{O}[u][\frac{u}{\pi}]\Big) \simeq \mathrm{Spec}(\mathcal{O}[u']), && u' = \frac{u}{\pi}, \end{aligned}$$

whose intersection is

$$D_+(u) \cap D_+(\pi) \simeq \mathrm{Spec}(\mathcal{O}[u, u'^{\pm 1}]) \simeq \mathrm{Spec}(\mathcal{O}[u'^{\pm 1}]).$$

Now, on the one hand, the Mayer-Vietoris sequence of K-theory implies immediately the following cartesian square of spectra:

$$\begin{array}{ccc} \mathrm{K}(\widetilde{X}) & \longrightarrow & \mathrm{K}(A^{\mathrm{nodal}}) \\ \downarrow & & \downarrow \\ \mathrm{K}(\mathcal{O}[u']) & \longrightarrow & \mathrm{K}(\mathcal{O}[u'^{\pm 1}]). \end{array}$$

By Theorem 1.2.2, we obtain an equivalence of spectra

$$\mathrm{NK}(\widetilde{X}) \xrightarrow{\simeq} \mathrm{NK}(A^{\mathrm{nodal}}) \tag{1.3.8}$$

and surjections on negative K-groups

$$\mathrm{K}_n(\widetilde{X}) \twoheadrightarrow \mathrm{K}_n(A^{\mathrm{nodal}}), \quad n < 0, \tag{1.3.9}$$

which are all isomorphisms. Indeed, the injectivity is clear if $n < 1$; at $n = -1$, the sequence extends to a long exact sequence

$$\mathrm{K}_0(A^{\mathrm{nodal}}) \oplus \mathrm{K}_0(\mathcal{O}[u']) \xrightarrow{\beta} \mathrm{K}_0(\mathcal{O}[u'^{\pm 1}]) \to \mathrm{K}_{-1}(\widetilde{X}) \to \mathrm{K}_{-1}(A^{\mathrm{nodal}}) \to 0, \tag{1.3.10}$$

where β is surjective since $\mathrm{K}_0(\mathcal{O}[u']) \simeq \mathrm{K}_0(\mathcal{O}) \simeq \mathbf{Z}$ by Theorem 1.2.2, and also that there is an equivalence $\mathrm{K}_0(\mathcal{O}[u'^{\pm 1}]) \simeq \mathrm{K}_0(\mathcal{O}) \oplus \mathrm{K}_{-1}(\mathcal{O}) \oplus \mathrm{NK}_0(\mathcal{O}) \oplus \mathrm{NK}_0(\mathcal{O}) \simeq \mathbf{Z}$ by Bass Fundamental Theorem.

On the other hand, the blow-up formula (1.3.7) provides canonical equivalences

$$\mathrm{K}(\widetilde{X}) \simeq \mathrm{K}(\mathcal{O}[u]) \oplus \mathrm{K}(\mathcal{O}/\pi),$$

$$\mathrm{K}(\widetilde{X}[T]) \simeq \mathrm{K}(\mathcal{O}[u][T]) \oplus \mathrm{K}(\mathcal{O}/\pi[T]),$$

where the latter holds because the blow-up commutes with flat base change. Then, by Theorem 1.2.2, we obtain an equivalence of spectra:

$$\mathrm{NK}(A^{\mathrm{nodal}}) \xrightarrow{\simeq} \mathrm{NK}(\mathcal{O}/\pi) \tag{1.3.11}$$

and isomorphisms of negative K-groups

$$\mathrm{K}_n(A^{\mathrm{nodal}}) \xrightarrow{\simeq} \mathrm{K}_n(\mathcal{O}/\pi), \quad n < 0. \tag{1.3.12}$$

Now one can conclude the proof of (1.3.3) and (1.3.4) in the case where $B = \mathcal{O}$.

Now we treat the case of general smooth $\mathcal{O}$-algebras B. Since blow-up commutes with flat base change, and in view of Theorem 1.2.2, the same arguments as above apply to any smooth base change of A^{nodal}, and this completes the proof. □

(B). *Formal completion.* The π-adic completion is a way of producing other sources of rings satisfying the ('pro"-)$\mathbf{A}^1$-homotopy invariance and negative K-group vanishing.

1.3.13 (K-theory and analytic isomorphism)**.** Let $A \to B$ be a (commutative) ring map. Let $s \in A$ be a non-zero-divisor in both A and B and such that $A/s^nA \xrightarrow{\simeq} B/s^nB$ for any $n \in \mathbf{N}$. Then $A \to B$ is a so-called *s-analytic isomorphism*. Then by Vorst–Weibel [Wei80, Theorem 1.3 (i)], we have a cartesian square of spectra

$$\begin{array}{ccc} \mathrm{K}(A) & \longrightarrow & \mathrm{K}(A[\frac{1}{s}]) \\ \downarrow & & \downarrow \\ \mathrm{K}(\widehat{A}) & \longrightarrow & \mathrm{K}(\widehat{A}[\frac{1}{s}]). \end{array} \tag{1.3.14}$$

Since $A[T] \to B[T]$ is then also an analytic isomorphism, the above applies to their polynomial extension with one variable, hence as a corollary [Wei80, Corollary 1.4], we have also a cartesian square of spectra

$$\begin{array}{ccc} \mathrm{NK}(A) & \longrightarrow & \mathrm{NK}(A[\frac{1}{s}]) \\ \downarrow & & \downarrow \\ \mathrm{NK}(\widehat{A}) & \longrightarrow & \mathrm{NK}(\widehat{A}[\frac{1}{s}]). \end{array} \tag{1.3.15}$$

1.3.16**.** Let A be a flat k°-algebra. Consider the π-adic completion $A \to \widehat{A}$, which is an analytic isomorphism along π (as $\widehat{A}$ is then also flat over k°).

Moreover, if $A[\frac{1}{\pi}]$ and $\widehat{A}[\frac{1}{\pi}]$ are both noetherian regular, by a classical result of Weibel [Wei13, Theorem V.6.3] they satisfy $\mathbf{A}^1$-homotopy invariance and negative K-group vanishing. As a result, we obtain

$$\mathrm{NK}(A) \xrightarrow{\simeq} \mathrm{NK}(\widehat{A}), \tag{1.3.17}$$

$$\mathrm{K}_n(A) \xrightarrow{\simeq} \mathrm{K}_n(\widehat{A}), \quad n < -1, \tag{1.3.18}$$

while at $n = -1$, there is an exact sequence

$$\mathrm{K}_0(A[\tfrac{1}{\pi}]) \oplus \mathrm{K}_0(\widehat{A}) \to \mathrm{K}_0(\widehat{A}[\tfrac{1}{\pi}]) \to \mathrm{K}_{-1}(A) \to \mathrm{K}_{-1}(\widehat{A}) \to 0. \tag{1.3.19}$$

(C). *Étale algebras over nodal algebras: a counterexample.* There exist counterexamples to the negative K-group vanishing in the semistable generality, even for semistable *affine curves*, due to the potential existence of loops in the dual graph; see Example 1.3.20.

1.3.20. **Example.** For semistable affine curves X, $\mathrm{K}_{-1}(X) \simeq H^1(\Gamma; \mathbf{Z})$, where Γ is the dual graph of X. One can construct one such X with non-trivial loops being present in Γ, so that $\mathrm{K}_{-1}(X) \not\simeq 0$.

1.4. **Standard semistable case.**

1.4.1. **Definition.** Let $\mathcal{O}$ be a valuation ring and $\pi \in \mathcal{O}$ be non-zero. For $d \in \mathbf{N}$, we define a *standard π-semistable $\mathcal{O}$-algebra* as

$$A_d^{\mathrm{sst}} \coloneqq \mathcal{O}[t_0, \dots, t_d]/(t_0 \dots t_d - \pi'),$$

where π' is a non-zero element of the radical $\sqrt{\pi\mathcal{O}}$. Since the context will be clear, we have omit the symbol π from the notation A_d^{sst}, and we will also simply call it a standard semistable $\mathcal{O}$-algebra, with π being implicit.

1.4.2. **Theorem.** *Let $\mathcal{O}$ be a valuation ring and $\pi \in \mathcal{O}$ which is non-zero and not a unit. For any smooth $\mathcal{O}$-algebra B, we have*

(i) $\mathrm{N}^\pi\mathrm{K}(A_d^{\mathrm{sst}} \otimes_{\mathcal{O}} B) \simeq 0$*;*
(ii) $\mathrm{K}_n(A_d^{\mathrm{sst}} \otimes_{\mathcal{O}} B) = 0$ *for* $n < 0$*;*
(iii) $\mathrm{NK}_n(A_d^{\mathrm{sst}} \otimes_{\mathcal{O}} B) = 0$ *for* $n \leq 0$.

Proof. Since $\mathrm{N}^{\pi}\mathrm{K}(-)$ is equivalent to $\mathrm{N}^{\pi'}\mathrm{K}(-)$ for any non-zero $\pi' \in \sqrt{\pi\mathcal{O}}$, we may assume that the π is equal to the π' appearing in the definition of A_d^{sst} (Definition 1.4.1).

We argue by induction on $d \geq 1$. (In fact, $d = 0$ corresponds to the regular case (Theorem 1.2.2).) The base case $d = 1$ has been treated with in Theorem 1.3.2 and Corollary 1.3.5. The (iii) would be a consequences of $\mathrm{NK}(A_d^{\mathrm{sst}} \otimes_{\mathcal{O}} B) \simeq \mathrm{NK}(B/\pi)$, which is stronger but may not be true.

Let $d \geq 2$. Consider the scheme $X \coloneqq \mathrm{Spec}(A_{d-1}^{\mathrm{sst}}[u])$ and the regular locus $Z \coloneqq \mathrm{Spec}(A_{d-1}^{\mathrm{sst}}[u]/(u, t_{d-1})) \simeq \mathrm{Spec}(\mathcal{O}/\pi[t_0, \dots, t_{d-2}])$. The closed immersion $Z \hookrightarrow X$ is quasi-smooth of codimension two. Let $\widetilde{X}$ be the blow-up $\mathrm{Bl}_Z X$. It is covered by two affine opens

$$D_+(u) \simeq \mathrm{Spec}\Big(A_{d-1}^{\mathrm{sst}}[u][\frac{t_{d-1}}{u}]\Big) \simeq \mathrm{Spec}(A_{d-1}^{\mathrm{sst}}[u, \bar{u}]/(u\bar{u} - t_{d-1})) \simeq \mathrm{Spec}(A_d^{\mathrm{sst}}), \qquad \bar{u} = \frac{t_{d-1}}{u},$$
$$D_+(t_{d-1}) \simeq \mathrm{Spec}\Big(A_{d-1}^{\mathrm{sst}}[u][\frac{u}{t_{d-1}}]\Big) \simeq \mathrm{Spec}(A_{d-1}^{\mathrm{sst}}[u']), \qquad u' = \frac{u}{t_{d-1}},$$

whose intersection is

$$D_+(u) \cap D_+(t_{d-1}) \simeq \mathrm{Spec}(A_{d-1}^{\mathrm{sst}}[u, u'^{\pm 1}]) \simeq \mathrm{Spec}(A_{d-1}^{\mathrm{sst}}[u'^{\pm 1}]).$$

Now, on the one hand, the Mayer-Vietoris sequence of K-theory implies immediately the following cartesian square of spectra:

$$\begin{array}{ccc} \mathrm{K}(\widetilde{X}) & \longrightarrow & \mathrm{K}(A_d^{\mathrm{sst}}) \\ \downarrow & & \downarrow \\ \mathrm{K}(A_{d-1}^{\mathrm{sst}}[u']) & \longrightarrow & \mathrm{K}(A_{d-1}^{\mathrm{sst}}[u'^{\pm 1}]), \end{array}$$

which induces a long exact sequence

$$\cdots \to \mathrm{K}_n(\widetilde{X}) \to \mathrm{K}_n(A_d^{\mathrm{sst}}) \oplus \mathrm{K}_n(A_{d-1}^{\mathrm{sst}}[u']) \to \mathrm{K}_n(A_{d-1}^{\mathrm{sst}}[u'^{\pm 1}]) \to \cdots,$$

and similarly, since blow-up commutes with flat base change, we have the corresponding cartesian diagram and the long exact sequence for NK in place of K. By induction hypotheses for $d - 1$, we obtain:

- an equivalence of spectra

$$\mathrm{N}^{\pi}\mathrm{K}(\widetilde{X}) \xrightarrow{\simeq} \mathrm{N}^{\pi}\mathrm{K}(A_d^{\mathrm{sst}}) \tag{1.4.3}$$

  by (i);
- surjections

$$\mathrm{NK}_n(\widetilde{X}) \twoheadrightarrow \mathrm{NK}_n(A_d^{\mathrm{sst}}), \quad n \leq 0 \tag{1.4.4}$$

  by (iii), which are isomorphisms for $n \leq -1$;
- surjections on negative K-groups

$$\mathrm{K}_n(\widetilde{X}) \twoheadrightarrow \mathrm{K}_n(A_d^{\mathrm{sst}}), \quad n < 0 \tag{1.4.5}$$

  by (ii), which are isomorphisms for $n < -1$.

The map (1.4.5) is an isomorphism at $n = -1$, too; indeed, the sequence extends to a long exact sequence

$$\mathrm{K}_0(A_d^{\mathrm{sst}}) \oplus \mathrm{K}_0(A_{d-1}^{\mathrm{sst}}[u']) \xrightarrow{\beta} \mathrm{K}_0(A_{d-1}^{\mathrm{sst}}[u'^{\pm 1}]) \to \mathrm{K}_{-1}(\widetilde{X}) \to \mathrm{K}_{-1}(A_d^{\mathrm{sst}}) \to 0, \tag{1.4.6}$$

where β is surjective since $\mathrm{K}_0(A_{d-1}^{\mathrm{sst}}[u']) \simeq \mathrm{K}_0(A_{d-1}^{\mathrm{sst}})$ by induction hypothesis (iii), and also that there is an equivalence $\mathrm{K}_0(A_{d-1}^{\mathrm{sst}}[u'^{\pm 1}]) \simeq \mathrm{K}_0(A_{d-1}^{\mathrm{sst}}) \oplus \mathrm{K}_{-1}(A_{d-1}^{\mathrm{sst}}) \oplus \mathrm{NK}_0(A_{d-1}^{\mathrm{sst}}) \oplus \mathrm{NK}_0(A_{d-1}^{\mathrm{sst}}) \simeq \mathrm{K}_{2-d}(A_{d-1}^{\mathrm{sst}})$ by Bass Fundamental Theorem, where $\mathrm{K}_{-1}(A_{d-1}^{\mathrm{sst}}) = 0$ and $\mathrm{NK}_0(A_{d-1}^{\mathrm{sst}}) = 0$ respectively by induction hypotheses (ii) and (iii).

On the other hand, the blow-up formula (1.3.7) provides canonical equivalences

$$\mathrm{K}(\widetilde{X}) \simeq \mathrm{K}(A_{d-1}^{\mathrm{sst}}[u]) \oplus \mathrm{K}(\mathcal{O}/\pi[t_0, \dots, t_{d-2}]),$$
$$\mathrm{NK}(\widetilde{X}) \simeq \mathrm{NK}(A_{d-1}^{\mathrm{sst}}[u]) \oplus \mathrm{NK}(\mathcal{O}/\pi[t_0, \dots, t_{d-2}]),$$

where the latter holds because the blowup commutes with flat base change. Then, by induction hypotheses, we obtain

- an equivalence of spectra:
$$\mathrm{N}^\pi\mathrm{K}(\widetilde{X}) \simeq 0;$$
- isomorphisms
$$\mathrm{NK}_n(\widetilde{X}) \simeq \mathrm{NK}_n(\mathcal{O}/\pi[t_0,\dots,t_{d-2}]), \quad n \leq 0;$$
- isomorphisms on negative K-groups
$$\mathrm{K}_n(\widetilde{X}) \xrightarrow{\simeq} \mathrm{K}_n(\mathcal{O}/\pi[t_0,\dots,t_{d-2}]), \quad n < 0.$$

Now one can conclude the proof of (i)–(iii) in the case where $B = \mathcal{O}$. The (i) is clear from above. For (iii), notice that for $n < 0$, $\mathrm{K}_n(\mathcal{O}/\pi[t_0,\dots,t_{d-2}]) \simeq \mathrm{K}_n((\mathcal{O}/\pi)_{\mathrm{red}}[t_0,\dots,t_{d-2}])$ by nil-invariance, which vanishes by Theorem 1.2.2; as its quotient (1.4.5), $\mathrm{NK}_n(A_d^{\mathrm{sst}})$ vanishes, too. Likewise for (ii), using (1.4.4). This proves the induction hypotheses for d and $B = \mathcal{O}$.

The case of general smooth $\mathcal{O}$-algebras B is the same, since blow-up commutes with flat base change, and in view of Theorem 1.2.2. This completes the induction step for d. □

1.5. **Semistable case.**

1.5.1 (Standard smooth algebras)**.** Let R be a ring. An R-algebra S is called *standard smooth* if there exists integers $n \geq c \geq 0$ and $f_1,\dots,f_c \in R[x_1,\dots,x_n]$ such that
$$S \simeq R[x_1,\dots,x_n]/(f_1,\dots,f_c)$$
and that the maximal minor $\det\left(\frac{\partial f_i}{\partial x_j}\right)_{1\leq i,j\leq c}$ is invertible in S.

For example, every étale R-algebra is standard smooth [Sta18, Lemma 00U9]. The class of standard smooth ring maps is stable under base change and composition [Sta18, Section 00T1].

1.5.2 (Lifting properties)**.** All étale ring maps lift along surjections of rings [Sta18, Lemma 04D1]; more generally, this holds for standard smooth ring maps (see the proof of [Sta18, Lemma 04B1]).

1.5.3. **Definition.** An $\mathcal{O}$-algebra is called *modelled on an $\mathcal{O}$-algebra C* if it is has an étale $C\otimes_{\mathcal{O}} B$-algebra structure for some standard smooth $\mathcal{O}$-algebra B. In particular, it is a standard smooth $\mathcal{O}$-algebra by 1.5.1.

1.5.4. **Example.** For example, any étale algebra over $\mathcal{O}[t_0,\dots,t_d]/(t_0\dots t_r - \pi')$ is modelled over $A_r^{\mathrm{sst}} = \mathcal{O}[t_0,\dots,t_r]/(t_0\dots t_r - \pi')$. Therefore, such $\mathcal{O}$-algebras for all non-zero $\pi' \in \sqrt{\pi\mathcal{O}} = k^{\circ\circ}$ provide an affine base of the Zariski topology on strictly π-semistable $\mathcal{O}$-schemes.

1.5.5. **Theorem.** *Let $\mathcal{O}$ be a valuation ring and $\pi \in \mathcal{O}$ which is non-zero and not a unit. For any $\mathcal{O}$-algebra A modelled on a standard π-semistable $\mathcal{O}$-algebra A_d^{sst}, we have*

(i) $\mathrm{N}^\pi\mathrm{K}(A) \simeq 0$ *in* Pro(Sp)*;*
(ii) $\mathrm{K}_n(A) = 0$ *for* $n < 0$*;*
(iii) $\mathrm{NK}_n(A) = 0$ *for* $n \leq 0$.

1.5.6. **Corollary.** *Let X be a strictly π-semistable $\mathcal{O}$-scheme. Then* $\mathrm{N}^\pi\mathrm{K}(X) \simeq 0$ *in* Pro(Sp).

Proof. The Pro(Sp)-valued presheaf $X \mapsto \mathrm{N}^\pi\mathrm{K}(X)$ satisfies Zariski descent on schemes. Hence we may assume X to be an affine scheme modelled over some A_d^{sst} (Example 1.5.4). Now the vanishing follows from Theorem 1.5.5 (i). □

1.5.7. **Remark** (Slight generalisation)**.** The semistability condition in Theorem 1.5.5 and Corollary 1.5.6 is not optimal. It is our first attempt in extending Theorem 1.2.2 to bad reduction with mild singularities. Here we discuss its potential generalisation.

(i) The semistability condition can be relaxed to a polystability condition. In fact, one checks easily that the same proof, but by induction on the poset of finite tuples of natural numbers, gives the same conclusion under polystability condition, i.e.:

(a) For any A algebra modelled on $A_{d_1}^{\mathrm{sst}} \otimes_{\mathcal{O}} \cdots \otimes_{\mathcal{O}} A_{d_n}^{\mathrm{sst}}$ for some tuple $(d_1, \ldots, d_n) \in \mathbf{N}^n$, we have $\mathrm{N}^\pi\mathrm{K}(A) \simeq 0$, $\mathrm{K}_n(A) = 0$ for $n < 0$, and $\mathrm{NK}_n(A) = 0$ for $n \leq 0$.

(b) For any strictly polystable (or equivalently, Zariski-locally modelled on finite products of standard semistable ones as above) scheme X over k°, we have $\mathrm{N}^\pi\mathrm{K}(X) \simeq 0$.

(ii) The induction process indicates that it be generalised to models with what we call (by ignorance of any literature related to it) *toric l.c.i. singularities*, i.e. models that are Zariski-locally modelled on an $\mathcal{O}$-algebra of the form

$$A_{\underline{M}} \coloneqq \mathcal{O}[z_1, \ldots, z_d, w_1, \ldots, w_d]/(z_j w_j - M_j, 1 \leq j \leq d),$$

where $d \in \mathbf{N}$ and M_j is a monomial on π and variables with index $\leq j-1$[6]. It is easily seen that $A_{\underline{M}}[\frac{1}{\pi}] \simeq \mathcal{O}[\frac{1}{\pi}][z_1^{\pm 1}, \ldots, z_d^{\pm 1}]$ is a standard toric chart over $\mathcal{O}[\frac{1}{\pi}]$.

Similar considerations apply to the formal case, but we will not repeat this remark and leave the interested to formulate the adequate statements.

Let us now prove Theorem 1.5.5.

Proof of Theorem 1.5.5. We argue by induction on $d \geq 0$. The case $d = 0$ corresponds to the regular case $A_0^{\mathrm{sst}} = \mathcal{O}$ (Theorem 1.2.2).

Let $d \geq 1$. We apply the same strategy as in the proof of Theorem 1.4.2.

First, let us show that, if A is base changed from an $\mathcal{O}$-algebra A_0 *modelled on* $A_{d-1}^{\mathrm{sst}}[u]$, then (i)–(iii) hold. Consider the scheme $X \coloneqq \mathrm{Spec}(A_0)$ and the regular locus $Z \coloneqq \mathrm{Spec}(A_0/(u, t_{d-1})) \simeq \mathrm{Spec}(\mathcal{O}/\pi[t_0, \ldots, t_{d-2}] \otimes_{A_{d-1}^{\mathrm{sst}}[u]} A_0)$. The blow-up $\widetilde{X} \coloneqq \mathrm{Bl}_Z X$ is covered by two affine opens

$$D_+(u) \simeq \mathrm{Spec}(A_0[u, \bar{u}]/(u\bar{u} - t_{d-1})) \simeq \mathrm{Spec}(A_d^{\mathrm{sst}} \otimes_{A_{d-1}^{\mathrm{sst}}[u]} A_0) = \mathrm{Spec}(A),$$

$$D_+(t_{d-1}) \simeq \mathrm{Spec}(A_{d-1}^{\mathrm{sst}}[u'] \otimes_{A_{d-1}^{\mathrm{sst}}[u]} A_0),$$

whose intersection is

$$D_+(u) \cap D_+(t_{d-1}) \simeq \mathrm{Spec}(A_{d-1}^{\mathrm{sst}}[u'^{\pm 1}] \otimes_{A_{d-1}^{\mathrm{sst}}[u]} A_0).$$

As in the proof of Theorem 1.4.2, by induction hypotheses, we have a cartesian square of spectra:

$$\begin{array}{ccc} \mathrm{K}(\widetilde{X}) & \longrightarrow & \mathrm{K}(A) \\ \downarrow & & \downarrow \\ \mathrm{K}(D_+(t_{d-1})) & \longrightarrow & \mathrm{K}(D_+(u) \cap D_+(t_{d-1})), \end{array}$$

and an equivalence

$$\mathrm{K}(\widetilde{X}) \simeq \mathrm{K}(A_0) \oplus \mathrm{K}(Z),$$

and the likewise for $\mathrm{NK}(-)$ in place of $\mathrm{K}(-)$. The coordinate rings of $D_+(t_{d-1})$ and $D_+(u) \cap D_+(t_{d-1})$ are both modelled on A_{d-1}^{sst} whenever A_0 is modelled on $A_{d-1}^{\mathrm{sst}}[u]$. Also the coordinate ring of Z is smooth over $\mathcal{O}/\pi$, hence Z_{red} is smooth over the valuation ring $(\mathcal{O}/\pi)_{\mathrm{red}}$. These allow to do induction on d: we obtain an equivalence

$$\mathrm{N}^\pi\mathrm{K}(A) \simeq 0,$$

and surjections

$$\mathrm{K}_n(A_0) \oplus \mathrm{K}_n(Z) \twoheadrightarrow \mathrm{K}_n(A), \qquad n < 0,$$

$$\mathrm{NK}_n(A_0) \oplus \mathrm{NK}_n(Z) \twoheadrightarrow \mathrm{K}_n(A), \qquad n \leq 0.$$

We conclude as before.

[6]This is the mixed characteristic analogue of Nakajima's classification, as described by Dais–Henk [DH03, Theorem 3.1], where here the ring $\mathcal{O}$ plays the role of $k[z_1]$ of *loc. cit.*

Moreover, the author feels compelled, in this large wave of declaration of AI usage in mathematical research, to point out that this reference has been found and informed to the author by a LLM model, although he feels that this claim resembles a disclosure ten (or twenty) years ago about the usage of Google in research. For this reason, the author will no longer reveal such functionality of AI in his future works.

Now we remove the base change assumption on A. Assume A to be *modelled* on A_d^{sst}, which we identify with $A_{d-1}^{\mathrm{sst}}[u,\bar{u}]/(u\bar{u}-t_{d-1})$. Choose a smooth $\mathcal{O}$-algebra B and an étale ring map $A_d^{\mathrm{sst}} \otimes_{\mathcal{O}} B \to A$. In order to simplify the notations, we set

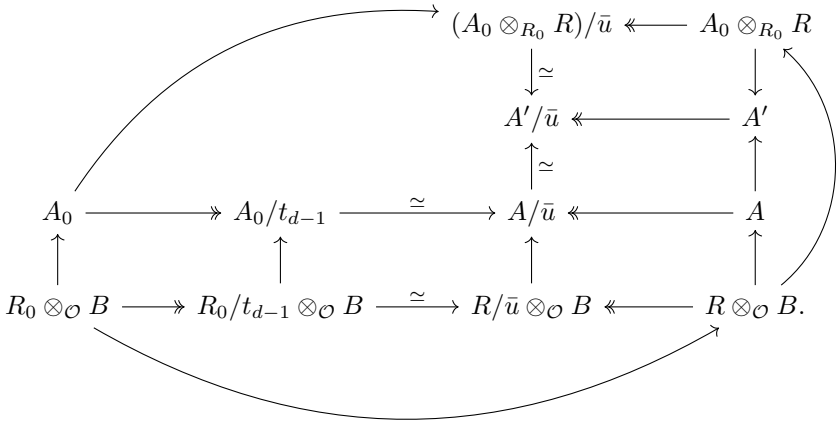
$$R_0 \coloneqq A_{d-1}^{\mathrm{sst}}[u], \quad R \coloneqq A_d^{\mathrm{sst}} = A_{d-1}^{\mathrm{sst}}[u,\bar{u}]/(u\bar{u}-t_{d-1}).$$

Previously, we established (i)–(iii) for A obtained as a base changed along $R_0 \to R$. Beware that $R_0 \to R$ is an isomorphism away from the locus $\{u=0\}$, over which the fibre is an affine line; hence most A do not arise in this way. One should not expect to prove a descent-to-R_0-type statement. Instead, we try to lift a reduction of this standard smooth algebra.

Observe that there is an isomorphism

$$\mathcal{O}/\pi[t_0,\dots,t_{d-2},u] = R_0/t_{d-1} \xrightarrow{\simeq} R/\bar{u}.$$

Consider the reduction $A/\bar{u}$; it is étale over $R_0/t_{d-1} \otimes_{\mathcal{O}} B$, hence it lifts to an étale $R_0 \otimes_{\mathcal{O}} B$-algebra A_0 by 1.5.2. By [Sta18, Lemma 04FV], can complete the following commutative diagram with an étale $R \otimes_{\mathcal{O}} B$-algebra A':



In fact, A' can be chosen to consist of idempotent component(s) of $(A_0 \otimes_{R_0} R) \otimes_{R \otimes_{\mathcal{O}} B} A$.

We want to know defects of $\mathbf{A}^1$-homotopy invariance about $\mathrm{K}(A)$. The top right two squares are Nisnevich squares and give rise to the following Nisnevich descent cartesian squares of spectra:

$$\begin{array}{ccc} \mathrm{K}(A_0 \otimes_{R_0} R) & \longrightarrow & \mathrm{K}((A_0 \otimes_{R_0} R)[\frac{1}{\bar{u}}]) \\ \downarrow & & \downarrow \\ \mathrm{K}(A') & \longrightarrow & \mathrm{K}(A'[\frac{1}{\bar{u}}]) \\ \uparrow & & \uparrow \\ \mathrm{K}(A) & \longrightarrow & \mathrm{K}(A[\frac{1}{\bar{u}}]) \end{array} \tag{1.5.8}$$

As $R[\frac{1}{\pi}] \simeq \mathcal{O}[\frac{1}{\pi}][t_0^{\pm 1},\dots,t_{d-2}^{\pm 1},u^{\pm 1}]$, which is smooth over $\mathcal{O}[\frac{1}{\pi}]$, which is stably coherent regular as $\mathcal{O}$ is stably coherent regular (see Lemma 1.2.5 and the proof of Proposition 1.2.9). Hence, being smooth over $R[\frac{1}{\pi}]$, the rings $A[\frac{1}{\bar{u}}]$, $A'[\frac{1}{\bar{u}}]$ and $(A_0 \otimes_{R_0} R)[\frac{1}{\bar{u}}]$ are all stably coherent regular as well by Lemma 1.2.5, so that their non-positive K-groups vanish and that their NK are contractible. Therefore, the left vertical maps in (1.5.8) become suitable equivalences, and we get

$$\mathrm{K}_n(A_0 \otimes_{R_0} R) \xrightarrow{\simeq} \mathrm{K}_n(A') \xleftarrow{\simeq} \mathrm{K}_n(A), \quad n<0,$$

$$\mathrm{NK}(A_0 \otimes_{R_0} R) \xrightarrow{\simeq} \mathrm{NK}(A') \xleftarrow{\simeq} \mathrm{NK}(A),$$

which, combined with the case that we have just seen (for the base changed R-algebra $A_0 \otimes_{R_0} R$ with A_0 modelled on R_0), proves (i)–(iii) for A. □

1.6. **Semistable formal case.** Let k be a nonarchimedean field. We are going to restrict ourselves from $\mathcal{O}$-algebras to k°-algebras.

1.6.1. Let $\mathfrak{X} = \mathrm{Spf}(A_0)$ be an admissible formal scheme of finite type over k°. Then $(A_0)_\eta \coloneqq A_0[\frac{1}{\pi}]$ is an affinoid Tate algebra over k. Here are some algebraic properties of $(A_0)_\eta$:

- it is a Noetherian Jacobson ring [Bos05, §3.1, Proposition 3];
- it is also regular if the rigid generic fibre $\mathfrak{X}_\eta$ is smooth over k (see for example [Duc09, Proposition 6.3]).

1.6.2. **Proposition.** *Let A_0 be an admissible k°-algebra and let $\widehat{A_0}$ be its π-adic completion. If A_0 is a strictly semistable k°-algebra, then $\mathrm{N}^\pi\mathrm{K}(\widehat{A_0}) \simeq 0$ in $\mathrm{Pro}(\mathrm{Sp})$.*

If moreover B_0 is a flat k°-algebra such that $B_0[\frac{1}{\pi}]$ is regular noetherian[7] and that $\widehat{B_0} \simeq \widehat{A_0}$, then $\mathrm{N}^\pi\mathrm{K}(B_0) \simeq 0$, too.

For example, B_0 could be a finite type algebra over a topologically finite type k°-algebra (e.g. a partial algebraisation of $\widehat{A_0}$).

Proof. This follows immediately from the formal-algebraic comparison 1.3.16 and Corollary 1.2.8. □

1.6.3. **Lemma.** *Any strictly semistable formal scheme $\mathfrak{X}$ is Zariski-locally of the form $\mathrm{Spf}(\widehat{A_0})$ for some strictly semistable k°-algebra A_0.*

Proof. We may assume that there is an étale morphism $f\colon \mathfrak{X} = \mathrm{Spf}(A_0) \to \mathrm{Spf}(k^\circ\langle t_0, \dots, t_d\rangle/(t_0 \dots t_r - \pi))$ of formal schemes, as in Definition 1.1.8. By topological invariance of the small étale site, f is uniquely determined by its reduction mod π, which we denote by $f_0\colon \mathrm{Spec}(A/\pi) \to \mathrm{Spec}(k^\circ/\pi[t_0, \dots, t_d]/(t_0 \dots t_r - \pi))$. This lifts to a (not necessarily unique) étale map $f'\colon A_0 \to \mathrm{Spec}(k^\circ/\pi[t_0, \dots, t_d]/(t_0 \dots t_r - \pi))$ (see 1.5.2). By the uniqueness of formal étale liftings, we have $\widehat{f'} \simeq f$. □

1.6.4 (Dagger condition)**.** Consider the following condition for an affinoid k-algebra A:

- $\left(\dagger_A^{(\mathrm{V})}\right)$ There exists a ring of definition A_0 of A, and a proper morphism of schemes $p\colon X \to \mathrm{Spec}(A_0)$ such that p is an isomorphism over $\mathrm{Spec}(A)$, that X is flat over k° and that its formal π-adic completion $\mathfrak{X}$ is a strictly semistable formal scheme over k°.

Beware that k is not necessarily discretely valued, so X is non-Noetherian and even not regular (in the non-Noetherian sense), contrary to its counterpart in the dagger condition in Kerz–Saito–Tamme's work[KST19]. We use the superscript "(V)" to indicate that it is our suggesstion of a generalisation of its friend in *loc. cit.* into the case of affinoid algebras "of type (V)".

Here are some basic properties of the rings involved:

(i) The scheme $X_\eta \xrightarrow{\simeq} \mathrm{Spec}(A)$ is Noetherian [Bos05, §3.1, Proposition 3]. It is also regular. Indeed, by [Hub96, (1.9.5), Proposition 1.9.6], using properness of p, the natural morphism $\mathfrak{X}_\eta \to \mathrm{Spa}(A, A^\circ) \times_{\mathrm{Spec}(A_0)} X$[8] of adic spaces (actually, adic spaces) is an isomorphism. Since p is an isomorphism over the generic fibre, we see that $\mathfrak{X}_\eta \xrightarrow{\simeq} \mathrm{Spa}(A, A^\circ)$. Then, since $\mathfrak{X}_\eta$ is smooth over k, so is $\mathrm{Spa}(A, A^\circ)$, hence A is also regular by 1.6.1. (This is needed for applying Proposition 1.6.2.)

(ii) For any affine open $\mathrm{Spf}(B) \subset \mathfrak{X}$, by semistability B_η is a smooth affinoid over k, hence it is Noetherian regular by 1.6.1.

[7] This noetherian regularity is needed in the argument of formal-algebraic below.

[8] The fibre product, as an adic space, always exists [Hub94, Proposition 3.8]. Instead of looking at X which are proper over $\mathrm{Spec}(A_0)$, one may attempt to compute $Y \coloneqq \mathrm{Spa}(A, A^\circ) \times_{\mathrm{Spec}(A_0)} \mathrm{Spec}(A)$: writing $A = A_0[X]/(X\pi - 1)$, then Y is the increasing union (gluing) of the rigid-analytic spectra of affinoid Tate k-algebras $A\langle \pi^k X\rangle/(X\pi - 1), k \geq 1$. But these are canonically isomorphic to A, so that $\mathrm{Spa}(A, A^\circ) \times_{\mathrm{Spec}(A_0)} \mathrm{Spec}(A) \xrightarrow{\simeq} \mathrm{Spa}(A, A^\circ)$. More generally, $\mathrm{Spa}(A, A^\circ) \times_{\mathrm{Spec}(A_0)} X_\eta \xrightarrow{\simeq} \mathrm{Spa}(A, A^\circ) \times_{\mathrm{Spec}(A_0)} X$ for any X locally of finite type over $\mathrm{Spec}(A_0)$.

1.6.5. **Remark.** In the definition of $\left(\dagger_A^{(\mathrm{V})}\right)$, we imposed the strict semistability on $\mathfrak{X}$, but as indicated in Remark 1.5.7, slight generalised variants could be considered.

Here is our analogue of [KST19, Corollary 4.7, Lemma 5.13].

1.6.6. **Definition.** For any topological k°-algebra R with π-adic topology, define the pro-spectra

$$\hat{\mathrm{N}}^\pi \mathrm{K}(R) \coloneqq \mathrm{fib}(\text{“}\lim_{T\mapsto \pi T}\text{”}\, \mathrm{K}(R\langle T\rangle) \to \mathrm{K}(R)),$$

where $R\langle T\rangle$ is the π-adic completion of $R[T]$.

1.6.7. **Proposition** ([KST19, Corollary 4.7, Lemma 5.13 (ii)])**.** *Let A be an affinoid k-algebra satisfying $\left(\dagger_A^{(\mathrm{V})}\right)$ together with a choice of ring of definition A_0. Then we have weak equivalences of pro-spectra:*

(i) $\mathrm{N}^\pi \mathrm{K}(A_0 \text{ on } \pi) \simeq 0$ *and* $\mathrm{N}^\pi \mathrm{K}(A_0) \simeq 0$,

(ii) $\hat{\mathrm{N}}^\pi \mathrm{K}(A_0 \text{ on } \pi) \simeq 0$.

Proof. We copy the proof of [KST19, Lemma 5.13 (ii)].

Let $p\colon X \to \mathrm{Spec}(A_0)$ be as in $\left(\dagger_A^{(\mathrm{V})}\right)$: X is flat over k°, its π-adic completion $\mathfrak{X}$ is a strictly semistable formal scheme over k°. (One can view X as a partial algebraisation of $\mathfrak{X}$.)

Notice that (ii) follows from (i). Indeed, by excision [TT07, Theorem 7.1], we have $\mathrm{N}^\pi \mathrm{K}(A_0 \text{ on } \pi) \xrightarrow{\simeq} \hat{\mathrm{N}}^\pi \mathrm{K}(A_0 \text{ on } \pi)$.

Now it remains to prove (i). First, by excision [TT07, Theorem 7.1], we get a commutative diagram of (pro-)spectra with exact rows:

$$\begin{array}{ccccc}
\mathrm{K}(\mathfrak{X} \text{ on } \pi) & \longrightarrow & \mathrm{K}(\mathfrak{X}) & \longrightarrow & \mathrm{K}(\mathfrak{X}_\eta) \\
\simeq\uparrow & & \uparrow & & \uparrow \\
\mathrm{K}(X \text{ on } \pi) & \longrightarrow & \mathrm{K}(X) & \longrightarrow & \mathrm{K}(X_\eta) \\
\uparrow & & \uparrow & & \simeq\uparrow \\
\mathrm{K}(A_0 \text{ on } \pi) & \longrightarrow & \mathrm{K}(A_0) & \longrightarrow & \mathrm{K}(A)
\end{array}$$

and its $\mathrm{N}^\pi \mathrm{K}(-)$ version. Now, observe the following:

- By the generalised pro-cdh descent [KST26, Theorem A][9] instead of pro-cdh descent for noetherian schemes [KST17, Theorem A], we have a *weak* equivalence[10] of pro-spectra:

$$\mathrm{N}^\pi \mathrm{K}(X) \xrightarrow{\simeq} \mathrm{N}^\pi \mathrm{K}(A_0).$$

This point corresponds exactly to $\mathrm{K}(-)$ satisfying the pro-descent condition (P1), cf. [KST19, Lemma 4.5]. However, the verification of the “regular vanishing” condition (P2) is unclear for X, so that the direct analogue of [KST19, Proposition 4.6, Corollary 4.7] does not immediately follows. That is why we need our treatment as follows in order to see that X behaves like K-regular schemes. Now let us continue with the observations:

- By smoothness of $\mathfrak{X}_\eta$ (from semistabiliy of $\mathfrak{X}$), we have an equivalence

$$\mathrm{N}^\pi \mathrm{K}(\mathfrak{X}_\eta) \simeq 0.$$

- By semistability of $\mathfrak{X}$, using (the first part of) Lemma 1.6.3 and Proposition 1.6.2 (and suitable descent as is needed to define $\mathrm{N}^\pi \mathrm{K}(\mathfrak{X})$, which I did not define actually...), we see that there is an equivalence

$$\mathrm{N}^\pi \mathrm{K}(\mathfrak{X}) \simeq 0.$$

[9]Using the topologically universal adhesiveness of (k°, π) [FK18, Chapter 0, Corollary 9.2.7], one can show that the admissible blow-up p, which is *a priori* only of finite type, is almost of finite presentation. This stronger finiteness assumption is imposed in [KST26, Theorem A]. Thanks George Tamme for pointing this out during an email exchange concerning the article [DYZ26].

[10]Due to pro-cdh descent, the whole result can only be weak equivalence, i.e. become an equivalence in $\mathrm{Pro}(\mathrm{Sp}^+)$, rather than an equivalence in $\mathrm{Pro}(\mathrm{Sp})$. This is the only place where *weak* equivalences get involved in the statement.

Therefore, we obtain a *weak* equivalence

$$\mathrm{N}^{\pi}\mathrm{K}(A_0 \text{ on } \pi) \simeq \mathrm{N}^{\pi}\mathrm{K}(X \text{ on } \pi) \simeq 0.$$

Now everything in the $\mathrm{N}^{\pi}\mathrm{K}$-version of the above diagram vanishes, except yet $\mathrm{N}^{\pi}\mathrm{K}(X) \xrightarrow{\simeq} \mathrm{N}^{\pi}\mathrm{K}(A_0)$[11]. Since $X_\eta \xrightarrow{\simeq} \mathrm{Spec}(A)$ is Noetherian and regular by 1.6.4 (i), we may apply (the second part of) Proposition 1.6.2 to conclude that there is a *weak* equivalence

$$\mathrm{N}^{\pi}\mathrm{K}(A_0) \simeq \mathrm{N}^{\pi}\mathrm{K}(X) \simeq 0.$$

This proves the second vanishing of (i), hence completes the proof. □

2. Dagger condition in Kerz–Saito–Tamme's work

We continue to work over k (i.e. the valuation ring $\mathcal{O}$ is taken to be k°).

We claim that the whole work of Kerz–Saito–Tamme [KST19] remains true for our new dagger condition $\left(\dagger_A^{(\mathrm{V})}\right)$ as long as we replace [KST19, Corollary 4.7, Lemma 5.13 (i)–(ii)] with our Proposition 2.1.1 below and Proposition 1.6.7 above.

This slight generalisation is intended, for example, for justifying that the étale representability of analytic K-theory in the rigid analytic motivic $\mathbf{A}^1$-homotopy category [DYZ26, Remark 4.1 (3), Remark 5.2] still hold for nonarchimedean field k which is not necessarily discretely valued[12]. This will be presented as a final part of this section.

This section is more like a survey and we claim no originality of results in this section.

2.1. Algebraic and continuous K-theory of Tate rings.

Let us record the following fact about the algebraic and the continuous K-theory of Tate rings, almost verbatim as in [KST19] as well as some basic results on analytic pro-homotopy invariance of their algebraic K-theory which follow from the previous section.

Let the continuous K-theory pro-spectrum be the one defined in [Mor16].

2.1.1. **Proposition** ([KST19, Lemma 5.13 (i)]). *Let A_0 be a topological algebra with π-adic topology. Then*

$$\hat{\mathrm{N}}^{\pi}\mathrm{K}^{\mathrm{cont}}(A_0) \coloneqq \mathrm{fib}(\text{“}\lim\text{”}_{T\mapsto\pi T}\mathrm{K}^{\mathrm{cont}}(A_0\langle T\rangle) \to \mathrm{K}^{\mathrm{cont}}(A_0))$$

is contractible in $\mathrm{Pro}(\mathrm{Sp})$.

Proof. The proof is elementary and the same as in *loc. cit.*: one can checked on each reduction A_0 momdulo a power of π: the pro-system $\left(\mathrm{K}(A_0\langle T\rangle/\pi^j)\right)_{T\mapsto\pi T}$ is equivalent to the constant A_0/π^j for any $j \in \mathbf{N}$. □

2.1.2. Let us start by observing that [KST19, Lemma 5.13 (iii)-(iv)] follow immediately from [KST19, Lemma 5.13 (ii)], whose generalised version here has been proved in Proposition 1.6.7. Therefore, the entire [KST19, Lemma 5.13] holds in our setting. For completeness, we record them together with their corollaries in below.

2.1.3. **Corollary** ([KST19, Lemma 5.13 (iii)-(iv), Proposition 5.14, Corollary 5.15, Proposition 5.16]). *Let A be an affinoid k-algebra satisfying $\left(\dagger_A^{(\mathrm{V})}\right)$.*

(i) *The canonical maps*

$$\hat{\mathrm{N}}^{\pi}\mathrm{K}(A) \leftarrow \hat{\mathrm{N}}^{\pi}\mathrm{K}(A_0) \to \mathrm{cofib}(\mathrm{K}(A_0) \to \text{“}\lim\text{”}_{T\mapsto\pi T}\mathrm{K}(A_0[\![T]\!]))$$

are weak equivalences of pro-spectra; the right one is an equivalence.

[11] This seems strange because in the proof of [KST19, Lemma 5.13] we deduce the vanishing of $\mathrm{N}^{\pi}\mathrm{K}(A_0 \text{ on } \pi)$ from the vanishing of $\mathrm{N}^{\pi}\mathrm{K}(A_0)$, or equivalently, from the vanishing of $\mathrm{N}^{\pi}\mathrm{K}(X)$; but here this vanishing is not automatic because we have to show first that $X_\eta \xrightarrow{\simeq} \mathrm{Spec}(A)$ is regular Noetherian!

[12] In [DYZ26, Remark 5.2], we should have imposed the discrete valuation since it relies on [DYZ26, Remark 4.1 (3)]. However, as can be seen from this note, the assertion of [DYZ26, Remark 5.2] remains correct as stated.

(ii) *The canonical map*
$$\mathrm{K}^{\mathrm{cont}}(A) \to \text{“}\lim_{T\mapsto\pi T}\text{”}\, \mathrm{K}^{\mathrm{cont}}(A\langle T\rangle)$$
is a weak equivalence of pro-spectra.

(iii) *The canonical map*
$$\mathrm{K}_0(A) \to \text{“}\lim_{T\mapsto\pi T}\text{”}\, \mathrm{K}_0(A\langle T\rangle)$$
is an isomorphism of pro-groups.

(iv) *We have vanishing* $R^i \lim_{T\mapsto\pi T} \pi_1(\hat{\mathrm{N}}\mathrm{K}(A)) = 0$ *for* $i \geq 0$.

Similar statements for finitely variables $T_1, \ldots, T_d$ *also hold.* □

2.2. **Analytic K-theory of adic rings and Tate rings.** By abuse of notation, let A be an adic ring with π-adic topology for some $\pi \in A$, or be a Tate ring with a pseudo-uniformiser π. The connective analytic K-theory $k^{\mathrm{an}}(A)$ à la Kerz–Saito–Tamme is defined by taking the inverse system, indexed by $j \in \mathbf{N}$, of the connective algebraic K-theory $k(-)$[13] of the geometric realisation of simplicial balls with radius $|\pi|^{-j}$. It is a particular case of the following construction: given any functor F from adic rings or Tate rings to the category $\mathrm{Sp}_{\geq 0}$ of *connective* spectra, we define a pro-spectrum via a system of connective spectra:
$$F^{\mathrm{an}}(A) \coloneqq \text{“}\lim_{j\in\mathbf{N}}\text{”}\, F(A\langle\Delta_{\pi^j}\rangle), \quad \text{where} \quad F(A\langle\Delta_{\pi^j}\rangle) \coloneqq \mathrm{colim}_{[n]\in\Delta^{\mathrm{op}}} F(A\langle\Delta^n_{\pi^j}\rangle).$$
Here, the connectiveness is technically needed for the convergence of the usual Postnikov spectral sequence computing the homotopy groups $\pi_*(F^{\mathrm{an}}(A))$ It does not matter whether the geometric realisation is taken in the category of connective spectra or in that of spectra, since the inclusion $\mathrm{Sp}_{\geq 0} \subset \mathrm{Sp}$ is a left adjoint to the connective cover functor $\tau_{\geq 0}$, hence preserves colimits ; the pro-limit $F^{\mathrm{an}}(A)$ is considered as a pro-spectrum.

As another example, taking F to be the connective continuous K-theory $k^{\mathrm{cont}} \coloneqq \tau_{\geq 1}\mathrm{K}^{\mathrm{cont}}$, the construction assigning F^{an} to F provides us the functor $k^{\mathrm{cont,an}}(-)$ for π-adic rings or Tate algebras.

For any $n \in \mathbf{N}$, let us denote $F_{\geq n}(A) \coloneqq \tau_{\geq n}F(A)$, and denote by $F_n(A)$ the Eilenberg-Maclane spectrum associated with the group $\pi_n F(A)$. There is a fibre sequence of spectra:
$$F_{\geq 1}(A) \to F(A) \to F_0(A).$$
This applies in particular to the connective K-theory, giving rise to the fibre sequence of spectra:
$$\mathrm{K}^{\mathrm{GL}}(A) \to k(A) \to K_0(A), \tag{2.2.1}$$
where we denote $\mathrm{K}^{\mathrm{GL}}(A) \coloneqq \mathrm{K}_{\geq 1}(A)$ for an ordinary ring A.

2.2.2. **Lemma** ([KST19, Lemma 6.4])**.** *Let A be an adic ring with π-adic topology or a Tate ring with pseudo-uniformiser π, there is a fibre sequence in* Pro(Sp)*:*
$$(F_{\geq 1})^{\mathrm{an}} \to F^{\mathrm{an}} \to \text{“}\lim_{j}\text{”}\, F_0(A\langle\Delta_{\pi^j}\rangle).$$
If the natural map $F_0(A) \to \text{“}\lim\text{”}_{\underline{T}\mapsto\pi\underline{T}} F_0(A\langle\underline{T}\rangle)$ is an isomorphism of pro-groups for any $\underline{T} = (T_1, \ldots, T_m)$ with any number of variables $m \in \mathbf{N}$, then there is a weak fibre sequence of pro-spectra
$$(F_{\geq 1})^{\mathrm{an}} \to F^{\mathrm{an}} \to F_0(A).$$

2.2.3. **Remark** (Summary of usage of $\left(\dagger^{(\mathrm{V})}_A\right)$)**.** Note that without the pro-homotopy invariance condition, $(F_{\geq 1})^{\mathrm{an}} \to F^{\mathrm{an}}$ may not be the connected cover. However, dagger condition $\left(\dagger^{(\mathrm{V})}_A\right)$ allows to deduce that the corresponding $(F_{\geq 1})^{\mathrm{an}} \to F^{\mathrm{an}}$ is indeed a connected cover. More precisely, $\left(\dagger^{(\mathrm{V})}_A\right)$ implies a multivariate pro-homotopy invariance that appears precedingly in either Proposition 1.6.7 or Corollary 2.1.3; consequently, Lemma 2.2.2 implies that the analytification preserves the corresponding connected cover. This will be the only use of $\left(\dagger^{(\mathrm{V})}_A\right)$ in the sequel.

[13]We define the connective algebraic K-theory $k(-)$ as the connective cover of the nonconnective one $\mathrm{K}(-)$.

Let us turn to specific analytic K-theoretic functors.

In the case of an adic ring, the connective continuous k-theory is itself already "analytic", as the lemma below shows. We will see that the analogue for Tate rings is conditional on $\left(\dagger_A^{(\mathrm{V})}\right)$.

2.2.4. **Lemma** ([KST19, Lemma 6.6, Proposition 6.7])**.** *Let A_0 be an adic ring with π-adic topology.*

(i) *The natural map $k^{\mathrm{cont}}(A_0) \to (k^{\mathrm{cont}})^{\mathrm{an}}(A_0)$ is an equivalence of pro-spectra.*

(ii) *The natural map $k^{\mathrm{an}}(A_0) \to (k^{\mathrm{cont}})^{\mathrm{an}}(A_0)$ is a weak equivalence of pro-spectra.*[14]

In particular, there is a natural weak equivalence of pro-spectra $k^{\mathrm{an}}(A_0) \simeq (k^{\mathrm{cont}})^{\mathrm{an}}(A_0)$. □

Let us turn to the case of Tate rings. As mentioned in Remark 2.2.3, this is where the dagger condition $\left(\dagger_A^{(\mathrm{V})}\right)$ comes into play.

2.2.5. **Definition.** Let A be a Tate ring and $A_0 \subset A$ be a subring of definition. Consider the connective k-theory spectrum with support

$$k(A_0 \text{ on } \pi) \coloneqq \tau_{\geq 0}\mathrm{fib}(K(A_0) \to K(A)).$$

This is the part where the $\left(\dagger_A^{(\mathrm{V})}\right)$ will affect. Mapping out to other K-theory spectra of adic rings, as well as considering the analytic construction, we have spectra

$$\begin{aligned}
\tilde{k}(A; A_0) &\coloneqq \mathrm{cofib}(k(A_0 \text{ on } \pi) \to k(A_0)),\\
\tilde{k}^{\mathrm{cont}}(A; A_0) &\coloneqq \mathrm{cofib}(k(A_0 \text{ on } \pi) \to k^{\mathrm{cont}}(A_0)),\\
\tilde{k}^{\mathrm{an}}(A; A_0) &\coloneqq \mathrm{cofib}(k^{\mathrm{an}}(A_0 \text{ on } \pi) \to k^{\mathrm{an}}(A_0)),\\
(\tilde{k}^{\mathrm{cont}})^{\mathrm{an}}(A; A_0) &\coloneqq \mathrm{cofib}(k^{\mathrm{an}}(A_0 \text{ on } \pi) \to (k^{\mathrm{cont}})^{\mathrm{an}}(A_0)).
\end{aligned}$$

They are all connective spectra; there are natural maps

$$\tilde{k}(A; A_0) \to k(A), \quad \tilde{k}^{\mathrm{an}}(A; A_0) \to k^{\mathrm{an}}(A).$$

2.2.6. **Lemma** ([KST19, Definition 6.11, Lemma 6.13])**.** *Let A be a Tate ring and $A_0 \subset A$ be a subring of definition.*

(i) *There are natural equivalences*

$$\tilde{k}_{\geq 1}(A; A_0) \simeq k_{\geq 1}(A) \simeq \mathrm{K}^{\mathrm{GL}}(A),$$

$$(\mathrm{K}^{\mathrm{GL}})^{\mathrm{an}}(A) \xrightarrow{\simeq} \tilde{k}^{\mathrm{an}}_{\geq 1}(A; A_0), \quad \tilde{k}^{\mathrm{cont}}_{\geq 1}(A; A_0) \xrightarrow{\simeq} \mathrm{K}^{\mathrm{cont}}_{\geq 1}(A; A_0) \simeq \mathrm{K}^{\mathrm{cont}}_{\geq 1}(A)$$

and natural maps

$$k_0^{\mathrm{an}}(A) \xrightarrow{\simeq} \pi_0(\text{“}\lim_j\text{”}\, K_0(A\langle\Delta_{\pi^j}\rangle)) \leftarrow \mathrm{K}_0(A) \leftarrow \mathrm{K}_0(A_0) \xrightarrow{\simeq} \tilde{k}_0(A; A_0) \xleftarrow{\simeq} \tilde{k}_0^{\mathrm{an}}(A; A_0).$$

(ii) *If A satisfies $\left(\dagger_A^{(\mathrm{V})}\right)$, then the above induces weak equivalences*

$$\tilde{k}^{\mathrm{an}}_{\geq 1}(A; A_0) \overset{\text{(i)}}{\xleftarrow{\simeq}} (\mathrm{K}^{\mathrm{GL}})^{\mathrm{an}}(A) \xrightarrow{\simeq} k^{\mathrm{an}}_{\geq 1}(A).$$

and an isomorphism of groups

$$k_0^{\mathrm{an}}(A) \simeq \mathrm{K}_0(A).$$

Proof. (i) It is the same proof as *loc. cit.* For the last isomorphism, one notice $K_0(A) \xrightarrow{\simeq} \tilde{k}_0(A; A_0)$ and the pro-homotopy invariance $\tilde{k}_0(A; A_0) \xrightarrow{\simeq} \text{“lim”}_j\, \tilde{k}_0(A\langle\Delta^m_{\pi^j}\rangle, A_0\langle\Delta^m_{\pi^j}\rangle)$ as in the proof of *loc. cit.* (using Proposition 2.1.1), which by Lemma 2.2.2 induces a weak fibre sequence

$$(\tilde{k}_{\geq 1})^{\mathrm{an}}(A; A_0) \to \tilde{k}^{\mathrm{an}}(A; A_0) \to \tilde{k}_0(A; A_0). \tag{2.2.7}$$

[14] We comment that the proof of (ii) uses the fibre sequence (2.2.1) and the completeness of the π-adic topology on A_0, see [KST19, Remark 6.10].

But notice that $\tilde{k}_{\geq 1}(A; A_0) \simeq \mathrm{K}^{\mathrm{GL}}(A)$, so the first term above is identified with $(\mathrm{K}^{\mathrm{GL}})^{\mathrm{an}}(A)$, which is connected, so that $\tilde{k}_0^{\mathrm{an}}(A; A_0) \to \tilde{k}_0(A; A_0)$ is an isomorphism and that there is a natural weak equivalence

$$\tilde{k}_{\geq 1}^{\mathrm{an}}(A; A_0) \xleftarrow{\simeq} (\tilde{k}_{\geq 1})^{\mathrm{an}}(A; A_0) \simeq (\mathrm{K}^{\mathrm{GL}})^{\mathrm{an}}(A).$$

(ii) Note that $K_0(A) \xrightarrow{\simeq}$ "lim"$_j \mathrm{K}_0(A\langle \Delta_{\pi^j}^m \rangle)$ is an isomorphism for any $m \in \mathbf{N}$; then by Corollary 2.1.3, the natural map $\mathrm{K}_0(A) \to$ "lim"$_j K_0(A\langle \Delta_{\pi^j} \rangle)$ is a weak equivalence of pro-spectra and hence by application of Lemma 2.2.2 there is a weak fibre sequence

$$(\mathrm{K}^{\mathrm{GL}})^{\mathrm{an}}(A) \to k^{\mathrm{an}}(A) \to \mathrm{K}_0(A), \tag{2.2.8}$$

the main point being that the third term, a priori only connective, is now concentrated in degree zero. The second isomorphism is now clear. For the first, observe that (2.2.8) induce a weak equivalences of spectra $(\mathrm{K}^{\mathrm{GL}})^{\mathrm{an}}(A) \xrightarrow{\simeq} k_{\geq 1}^{\mathrm{an}}(A)$. □

In order to define a nonconnective K-theory, it is natural to apply an analytic Bass construction $(-)^B$ [KST19, §4.4].

Kerz–Saito–Tamme define the *nonconnective analytic* K*-theory* of a Tate ring A as the analytic Bass delooping of $(\mathrm{K}^{\mathrm{GL}})^{\mathrm{an}}(A)$, namely

$$\mathrm{K}^{\mathrm{an}}(A) \coloneqq ((\mathrm{K}^{\mathrm{GL}})^{\mathrm{an}})^B(A).$$

It is compared below with another candidate, $(k^{\mathrm{an}})^B(A)$, which yields a weakly equivalent construction if A satisfies the condition $\left(\dagger_A^{(\mathrm{V})}\right)$.

2.2.9. **Lemma** ([KST19, Lemma 6.18])**.** *The natural maps* $\mathrm{K}^{\mathrm{GL}}(A) \to \tilde{k}^{\mathrm{an}}(A; A_0) \to k^{\mathrm{an}}(A)$ *induce via the analytic Bass construction natural maps*

$$\mathrm{K}^{\mathrm{an}}(A) \to (\tilde{k}^{\mathrm{an}})^B(A; A_0) \to (k^{\mathrm{an}})^B(A).$$

(i) *The first arrow is a weak equivalence of pro-spectra.*
(ii) *If* A *satisfies* $\left(\dagger_A^{(\mathrm{V})}\right)$*, then the second arrow is an equivalence.*

Proof. The proof *loc. cit.* works verbatim; under the condition $\left(\dagger_A^{(\mathrm{V})}\right)$ one applies Lemma 2.2.6 (ii). □

However, $\mathrm{K}^{\mathrm{an}}(A)$ has better formal properties such as analytic descent [KST23].

Now, let us compare the analytic K-theory with the continuous K-theory.

Recall that the continuous K-theory $\mathrm{K}^{\mathrm{cont}}(A)$ of a Tate ring A can be recovered from $\tilde{k}^{\mathrm{cont}}(A; A_0)$ via the analytic Bass construction:

2.2.10. **Lemma** ([KST19, Lemma 6.14])**.** *We have an equivalence of pro-spectra* $(\tilde{k}^{\mathrm{cont}})^B(A; A_0) \xrightarrow{\simeq} \mathrm{K}^{\mathrm{cont}}(A; A_0) \simeq \mathrm{K}^{\mathrm{cont}}(A)$. □

2.2.11. **Lemma** ([KST19, Lemma 6.12])**.** *Consider natural maps:*

$$\tilde{k}^{\mathrm{cont}}(A; A_0) \to (\tilde{k}^{\mathrm{cont}})^{\mathrm{an}}(A; A_0) \leftarrow \tilde{k}^{\mathrm{an}}(A; A_0). \tag{2.2.12}$$

(i) *The right map is always a weak equivalence.*
(ii) *If* A *satisfies* $\left(\dagger_A^{(\mathrm{V})}\right)$*, then the left map is also a weak equivalence.*

Proof. (i) This is clear from definition and Lemma 2.2.4.

(ii) The part involving A_0 without support is also clear from definition and Lemma 2.2.4, without any further restriction on A_0. For the comparison of the parts with support: the natural map $k(A_0 \text{ on } \pi) \to k^{\mathrm{an}}(A_0 \text{ on } \pi)$ is an equivalence under the condition $\left(\dagger_A^{(\mathrm{V})}\right)$ by Proposition 1.6.7 (ii) and a computation of homotopy groups via spectral sequence. □

Applying the analytic Bass construction, we obtain the desired comparison of various nonconnective pro-spectra:

2.2.13. **Corollary** ([KST19, Thm. 6.19, Cor. 6.20])**.** *Consider the natural maps, as obtained by applying the analytic Bass construction to (2.2.12) (see also Lemma 2.2.10 and Lemma 2.2.9 for the identifications used below):*

$$K^{\mathrm{cont}}(A) \to ((\tilde{k}^{\mathrm{cont}})^{\mathrm{an}})^B(A; A_0) \leftarrow K^{\mathrm{an}}(A).$$

(i) *The right map is always a weak equivalence.*

(ii) *If A satisfies $\left(\dagger_A^{(\mathrm{V})}\right)$, then the left map is also a weak equivalence.*

Furthermore, consider the natural maps

$$\mathrm{K}^{\mathrm{an}}(A) \to (k^{\mathrm{an}})^B(A) \leftarrow k^{\mathrm{an}}(A).$$

(iii) *If A satisfies $\left(\dagger_A^{(\mathrm{V})}\right)$, then they induce a weak equivalence of pro-spectra*

$$\mathrm{K}^{\mathrm{an}}_{\geq 1}(A) \simeq \mathrm{K}^{\mathrm{cont}}_{\geq 1}(A) \simeq k^{\mathrm{an}}_{\geq 1}(A)$$

and an isomorphism of pro-groups

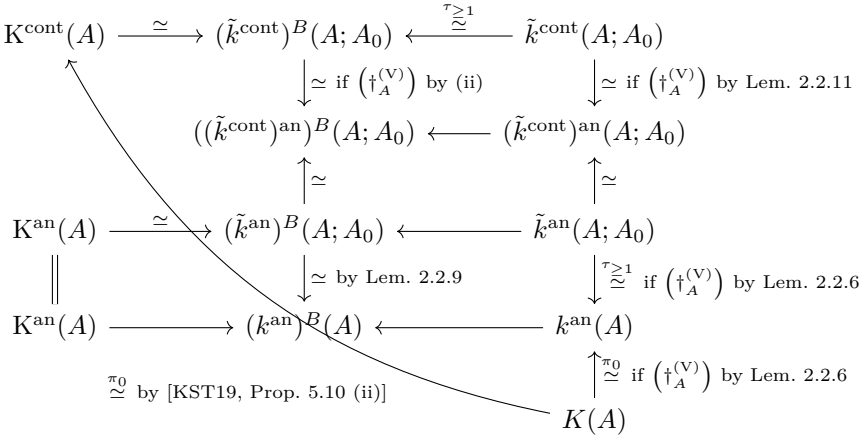
$$k_0^{\mathrm{an}}(A) \simeq \mathrm{K}_0(A) \simeq \mathrm{K}_0^{\mathrm{cont}}(A) \simeq \mathrm{K}_0^{\mathrm{an}}(A).$$

In particular, we have a weak equivalence of pro-spectra

$$k^{\mathrm{an}}(A) \xrightarrow{\simeq} K^{\mathrm{an}}_{\geq 0}(A)$$

Proof. The (i)–(ii) follow immediately from the previous lemma.

For (iii), consider the following commutative diagram:



which summarises well the comparisons that we have made so far. Now, on the one hand, looking at the weak equivalences for $\tau_{\geq 1}$, one obtains immediately that $\mathrm{K}^{\mathrm{an}}_{\geq 1}(A) \simeq k^{\mathrm{an}}_{\geq 1}(A)$. Now consider On the other hand, it is a general fact that the continuous K-theory of Tate rings satisfies the Bass fundamental theorem and does not complicate the $\pi_0(-)$, i.e. we have $\mathrm{K}_0(A) \xrightarrow{\simeq} \mathrm{K}_0^{\mathrm{cont}}(A)$; see [KST19, Prop. 5.10]; therefore, the arrow from the bottom right to the top left is an isomorphism on $\pi_0(-)$. One sees then that $k_0^{\mathrm{an}}(A) \xleftarrow{\simeq} \mathrm{K}_0(A) \xrightarrow{\simeq} \mathrm{K}_0^{\mathrm{cont}}(A) \xrightarrow{\simeq} \mathrm{K}_0^{\mathrm{an}}(A)$, □

2.3. **Analytic** KV**-theory of Tate rings.** The Karoubi–Villamayor K-theory of a ring R, which maps from the algebraic K-theory, is a simpler variant of the latter. By definition, we have $\mathrm{KV}_n(A) \coloneqq \pi_{n-1}(\mathrm{GL}(A[\Delta^\bullet])) = \pi_n(\mathrm{BGL}(A[\Delta^\bullet]))$ for $n \geq 1$. It has basic properties that $\mathrm{KV}_1(A) = \mathrm{GL}(A)/U(A)$, where $U(A)$ is the subgroup of unipotent matrices, and that $\mathrm{KV}_n(A)$ is the KV_1 of the path algebra $\Omega^{n-1}A$ for $n \geq 1$. Moreover, it is always homotopy invariant. Recall that $\mathrm{U}(A)$ consists of $I + N$ where N is nilpotent.

In the analytic theory where A is a Tate ring with pseudo-uniformiser, as usual, the simplex each m-simplex $A[\Delta^m]$ over A is replaced by $A\langle\Delta^m_{\pi^j}\rangle$, controlled by "radii" $|\pi|^{-j}$ indexed by $j \in \mathbf{N}$.

2.3.1. **Definition** ([KST19, Def. 7.1])**.** The *analytic* KV*-theory* of A is defined as the pointed pro-connected space

$$\mathrm{KV}^{\mathrm{an}}(A) \coloneqq \text{“}\lim_j\text{”}\, \mathrm{BGL}(A\langle\Delta_{\pi^j}\rangle),$$

whose n-th homotopy pro-group $\mathrm{KV}^{\mathrm{an}}_n(A)$ ($n \geq 1$) are called the *analytic* KV*-groups* of A.

It is well-defined and independent of the choice of π, and we have isomorphisms of pro-groups:

$$\mathrm{KV}^{\mathrm{an}}_n(A) \simeq \text{“}\lim_j\text{”}\, \pi_{n-1}(\mathrm{GL}(A\langle\Delta_{\pi^j}\rangle)), \quad n \geq 1.$$

Since any Tate ring is a normed ring, $\mathrm{U}(A)$ has finer variants $\mathrm{U}(A)_{\pi^j}$ indexed by $j \in \mathbf{N}$, which consists of $I + N$ with $\lim_{n\to\infty} \|N^n\|\left|\pi^j\right| = 0$. The $\mathrm{KV}^{\mathrm{an}}_1(A)$ can be expressed in terms of these subgroups:

$$\mathrm{KV}^{\mathrm{an}}_1(A) \simeq \text{“}\lim_j\text{”}\, \mathrm{GL}(A)/\mathrm{U}(A)_{\pi^j}.$$

One can compare $\mathrm{KV}^{\mathrm{an}}(A)$ with $\mathrm{K}^{\mathrm{an}}(A)$. The simplicial ring $A\langle\Delta^\bullet_{\pi^j}\rangle$ being connected, $\mathrm{BGL}(A\langle\Delta_{\pi^j}\rangle)$ is weakly equivalent to the infinite loop space $\Omega^\infty\mathrm{K}^{\mathrm{GL}}(A\langle\Delta_{\pi^j}\rangle)$ as pointed connected spaces, hence there is a weak equivalence of pointed pro-connected spaces:

$$\mathrm{KV}^{\mathrm{an}}(A) \simeq \text{“}\lim_j\text{”}\, \Omega^\infty\mathrm{K}^{\mathrm{GL}}(A\langle\Delta_{\pi^j}\rangle) \simeq \Omega^\infty(\mathrm{K}^{\mathrm{GL}})^{\mathrm{an}}(A). \tag{2.3.2}$$

2.3.3. **Lemma** ([KST19, Lem. 7.5])**.** *Let A be a Tate ring satisfying $\left(\dagger^{(\mathrm{V})}_A\right)$, then we have natural weak equivalences of pointed pro-connected spaces:*

$$\mathrm{KV}^{\mathrm{an}}(A) \xrightarrow{\simeq} \Omega^\infty k^{\mathrm{an}}_{\geq 1}(A) \xrightarrow{\simeq} \Omega^\infty\mathrm{K}^{\mathrm{an}}_{\geq 1}(A).$$

Proof. It suffices to append the connected parts of Lemma 2.2.6 (ii) and then those of Corollary 2.2.13 (iii) to the unconditional comparison (2.3.2). □

2.4. **Application: representability of analytic** K**-theory.** Let us provide a version of the representablity theorem [DYZ26, Thm. B]. Let k be a nonarchimedean field. Let $\mathrm{RigH}(k, \mathrm{Pro}^\omega(\mathrm{Spc}^+_*))$ be the unstable $\mathbf{A}^1$-homotopy category of rigid spaces over k à la Dahlhausen-Yaylali [DY26] with coefficients in the ∞-category of countably indexed pro-spaces. There is a localisation functor $\mathbf{L}_{\mathrm{mot}}$ from $\mathrm{Pro}^\omega(\mathrm{Spc}^+_*)$-valued presheaves on the category SmRig_k of smooth rigid spaces over k to $\mathrm{RigH}(k, \mathrm{Pro}^\omega(\mathrm{Spc}^+_*))$. In particular, one get an object $\mathbf{L}_{\mathrm{mot}}(\mathbf{Z} \times \mathrm{BGL})$ of the latter category. We may also consider variants with coefficients in $\mathrm{Pro}^\omega(\mathrm{Sp})$, $\mathrm{Pro}^\omega(\mathrm{Sp}^+)$, $\mathrm{Pro}^\omega(\mathrm{Spc}_*)$, etc.; a morphism in any of these categories without the $+$-sign is a weak equivalence if and only if it becomes a an equivalence in the corresponding category with the $+$-sign. On the other hand, for affinoid rigid spaces over k, the functor $\mathrm{Spa}(A, A^+) \mapsto \mathrm{K}^{\mathrm{an}}(A)$ satisfies analytic descent [DYZ26, Lem. 4.23] hence extends uniquely to a sheaf on SmRig_k with values in $\mathrm{Pro}^\omega(\mathrm{Sp}^+)$; it is also $\mathbf{A}^1$-homotopy invariant [KST23, Prop. 4.4]. Therefore, we obtain an object $\mathrm{K}^{\mathrm{an}} \in \mathrm{RigH}(k, \mathrm{Pro}^\omega(\mathrm{Sp}^+))$, and by applying right adjoints, one obtains an object $\Omega^\infty\mathrm{K}^{\mathrm{an}}_{\geq 0} \in \mathrm{RigH}(k, \mathrm{Pro}^\omega(\mathrm{Spc}^+_*))$.

2.4.1. **Theorem** ([DYZ26, Thm. 5.1])**.** *Assume that every smooth rigid space over k is locally, for the analytic topology, of the form $\mathrm{Spa}(A, A^+)$ with A satisfying $\left(\dagger^{(\mathrm{V})}_A\right)$. Then there is a canonical equivalence $\Omega^\infty\mathrm{K}^{\mathrm{an}}_{\geq 0} \simeq \mathbf{L}_{\mathrm{mot}}(\mathbf{Z} \times \mathrm{BGL})$ in $\mathrm{RigH}(k, \mathrm{Pro}^\omega(\mathrm{Spc}^+_*))$.*

Proof. Verbatim as *loc. cit.*, since we have extended the validity of results of [**?**] to the setting where k is not necessarily discretely valued. □

Let $\mathbf{L}_{\text{ét}}$ denote the étale sheafification of a presheaf on SmRig_k with values in $\mathrm{Pro}^\omega(\mathrm{Spc}^+_*)$ (*resp.* in $\mathrm{Pro}^\omega(\mathrm{Sp}^+)$).

2.4.2. **Theorem** ([DYZ26, Rem. 5.2], [Zho25, Thm. B.3.3])**.** *There is a canonical equivalence $\Omega^\infty\tau_{\geq 0}\mathbf{L}_{\text{ét}}\mathrm{K}^{\mathrm{an}} \simeq \mathbf{L}_{\text{ét}}\mathbf{L}_{\mathbf{A}^1}(\mathbf{Z} \times \mathrm{BGL})$ in $\mathrm{Shv}_{\text{ét}}(\mathrm{SmRig}_k, \mathrm{Pro}^\omega(\mathrm{Spc}^+_*))$.*

Proof. Étale locally, the condition $\left(\dagger^{(\mathrm{V})}_A\right)$ is verified by Temkin's altered local uniformisation [Tem17, Cor. 3.3.2]. The proof then goes almost verbatim, with difference lying in the coefficients. □

References

[AMM22] Ben Antieau, Akhil Mathew, and Matthew Morrow. The K-theory of perfectoid rings. *Documenta Mathematica*, 27:1923–1951, 2022. doi:10.4171/DM/X22.

[Bos05] Siegfried Bosch. *Lectures on formal and rigid geometry*. Springer, 2005. doi:10.1007/978-3-319-04417-0.

[BS22] Sourayan Banerjee and Vivek Sadhu. K-theory of Prüfer domains. *Archiv der Mathematik*, 118(5):465–470, 2022. doi:10.1007/s00013-022-01721-5.

[DH03] Dimitrios I. Dais and Martin Henk. On the Equations Defining Toric l.c.i.-singularities. *Transactions of the American Mathematical Society*, 355(12):4955–4984, 2003. doi:10.1090/S0002-9947-03-03218-5.

[Duc09] Antoine Ducros. Les espaces de Berkovich sont excellents. *Annales de l'Institut Fourier*, 59(4):1443–1552, 2009. doi:10.5802/aif.2470.

[DY26] Christian Dahlhausen and Can Yaylali. Towards $\mathbb{A}^1$-homotopy theory of rigid analytic spaces, 2026. arXiv:2407.09606v3.

[DYZ26] Christian Dahlhausen, Can Yaylali, and Yicheng Zhou. Representability of continuous K-theory in rigid analytic motivic $\mathbb{A}^1$-homotopy theory, 2026. arXiv:2608.06209v1.

[FK18] Kazuhiro Fujiwara and Fumiharo Kato. *Foundations of Rigid Geometry I*, volume 7 of *Monographs in Mathematics*. EMS, 2018. doi:10.4171/135.

[GL24] Ning Guo and Fei Liu. Purity and quasi-split torsors over prüfer bases. *Journal de l'École polytechnique Mathématiques*, 11:187–246, January 2024. doi:10.5802/jep.253.

[Gla89] Sarah Glaz. *Commutative Coherent Rings*, volume 1371 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, Heidelberg, 1989. doi:10.1007/BFb0084570.

[Hub94] Roland Huber. A generalization of formal schemes and rigid analytic varieties. *Mathematische Zeitschrift*, 217(1):513–551, 1994. doi:10.1007/BF02571959.

[Hub96] Roland Huber. *Étale cohomology of rigid analytic varieties and adic spaces*, volume E30. Wiesbaden: Vieweg, 1996. doi:10.1007/978-3-663-09991-8.

[Kha20] Adeel A. Khan. Algebraic K-theory of quasi-smooth blow-ups and cdh descent. *Annales Henri Lebesgue*, 3:1091–1116, September 2020. doi:10.5802/ahl.55.

[KM21] Shane Kelly and Matthew Morrow. K-theory of valuation rings. *Compositio Mathematica*, 157(6):1121–1142, 2021. doi:10.1112/S0010437X21007119.

[KST17] Moritz Kerz, Florian Strunk, and Georg Tamme. Algebraic K-theory and descent for blow-ups. *Inventiones mathematicae*, 211(2):523–577, August 2017. doi:10.1007/s00222-017-0752-2.

[KST19] Moritz Kerz, Shuji Saito, and Georg Tamme. K-theory of non-Archimedean rings. I. *Doc. Math.*, 24:1365–1411, 2019. doi:10.25537/dm.2019v24.1365-1411.

[KST23] Moritz Kerz, Shuji Saito, and Georg Tamme. K-Theory of Non-Archimedean Rings II. *Nagoya Mathematical Journal*, 251:669–685, 2023. doi:10.1017/nmj.2023.4.

[KST26] Shane Kelly, Shuji Saito, and Georg Tamme. On pro-cdh descent on derived schemes. *Geom. Topol.*, 30(1):337–372, 2026. doi:10.2140/gt.2026.30.337.

[Mor16] Matthew Morrow. Pro cdh-descent for cyclic Homology and K-theory. *Journal of the Institute of Mathematics of Jussieu*, 15(3):539–567, 2016. doi:10.1017/S1474748014000413.

[Sta18] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu, 2018.

[Tem17] Michael Temkin. Altered local uniformization of Berkovich spaces. *Israel Journal of Mathematics*, 221(2):585–603, 2017. doi:10.1007/s11856-017-1557-0.

[Tho93] R.W. Thomason. Les K-groupes d'un schéma éclaté et une formule d'intersection excédentaire. *Inventiones mathematicae*, 112(1):195–216, 1993. doi:10.1007/BF01232430.

[TT07] Robert W. Thomason and Thomas Trobaugh. Higher algebraic k-theory of schemes and of derived categories. In Pierre Cartier, Luc Illusie, Nicholas M. Katz, Gérard Laumon, Yuri I. Manin, and Kenneth A. Ribet, editors, *The Grothendieck Festschrift: A Collection of Articles Written in Honor of the 60th Birthday of Alexander Grothendieck*, pages 247–435. Birkhäuser Boston, Boston, MA, 2007. doi:10.1007/978-0-8176-4576-2_10.

[Wei80] Charles A. Weibel. k-theory and analytic isomorphisms. *Inventiones mathematicae*, 61(2):177–197, 1980. doi:10.1007/BF01390120.

[Wei13] Charles A. Weibel. *The K-book. An introduction to algebraic K-theory*, volume 145 of *Grad. Stud. Math.* Providence, RI: American Mathematical Society (AMS), 2013. doi:10.1090/gsm/145.

[Zho25] Yicheng Zhou. *Sur les cohomologies étale et syntomique pour les variétés analytiques p-adiques arithmétiques*. Theses, Sorbonne Université, July 2025. URL: https://theses.hal.science/tel-05238283.

Université Paris-Saclay, CNRS, Laboratoire de Mathématiques d'Orsay, 91405, Orsay, France

Email address: yicheng.zhou@universite-paris-saclay.fr